Subsidizing the Distribution Channel: Donor Funding to Improve the Availability of Products with Positive Externalities

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Abstract

Large populations in the developing world lack adequate access to products, such as essential medicines, whose use confers positive externalities. Because of the critical role played by the private-sector distribution channel, donors (e.g., the Global Fund) are beginning to devote substantial resources to fund subsidies that encourage the channel to improve the availability of these products. A key question for a donor is whether it should subsidize the purchases or sales of the private channel. We find the answer depends crucially on whether consumers are homogeneous or heterogeneous in their valuation of the product. For the case of heterogeneous consumers' valuations, we provide evidence that subsidizing sales leads to greater expected donor utility, consumption, and social welfare. When consumers' valuations are homogeneous, we establish the opposite conclusion: subsidizing purchases leads to greater expected donor utility, consumption, and social welfare.

Key words: global health supply chains; developing country supply chains; subsidies; externalities

1 Introduction

In developing countries, products for health, food and other basic needs are distributed through both a government-run system and a private-sector distribution channel. In many countries, the private distribution channel plays a much more important role in providing access to these products (Bustreo et al. 2003, International Finance Corporation 2007), in part because of the limited geographic reach of the government-run outlets. In some countries, over 70% of the population accesses products of this type via the private distribution channel (Sabot et al. 2008). Private distribution channels have been remarkably successful in distributing consumer products such as soft drinks in poor and remote parts of the world (Yadav et al. 2010) and thus offer enormous possibilities for expanding access to products with a societal benefit. However, for products such as medicines and vaccines, where the benefits of consuming the product do not accrue solely to the consumer, private supply chains fail in providing high levels of availability of the product (Cameron et al. 2009). For example, a study in Senegal found that only 8% of the private shops had the recommended drug for malaria in stock and in those shops the stocking level was very low (Sabot et al. 2008).

There are a large number of products whose use benefits both the individual using the product and society at large (i.e., the products possess *positive externalities*). Examples include mosquito nets, condoms, and various medicines (e.g., oral rehydration salts for treatment of diarrhea) and vaccines. Among products with positive externalities, many have limited shelf lives. Notable examples include vaccines such as those for human papillomavirus and hepatitis B, Artemisinin Combination Therapies (ACTs) for malaria treatment, and some fortified foods. Some of these products, such as malaria drugs, have a short demand season due to the seasonal peaks observed in malaria incidence coinciding with the rainy season. Other products, such as vaccines and fortified foods, are distributed in campaigns which are run for a fixed duration. The disease incidence, transmission intensity, peak amplitudes and lengths of the disease season, and food yield and famine can vary significantly from year to year. Consequently, the demand for malaria drugs, vaccines and other specific health products is highly uncertain, especially at the level of the small geographic region (e.g., a village) which an individual retailer serves.

For many products with positive externalities, especially medicines, vaccines and fortified foods, bilateral donors such as the U.S. Government, multilateral agencies such as the World Bank and the Global Fund to Fight AIDS, Tuberculosis and Malaria, large non-governmental organizations such as the Clinton Health Access Initiative, and private philanthropic organizations such as the Bill and Melinda Gates Foundation, intervene to improve access to populations in poor countries. Because of the important role played by the private distribution channel, a primary way donors seek to achieve this objective is by designing and then funding product subsidies that encourage the channel to make decisions (e.g., stocking and/or pricing decisions) that increase the availability of the product to end consumers.

A key question in designing a subsidy is how it should be administered in the supply chain. One option is to administer the subsidy upstream: reduce the cost of each unit the channel acquires via a *purchase subsidy*. A second option is to administer the subsidy downstream: increase the revenue for each unit the channel sells via a *sales subsidy*. Both forms of donor-funded subsidies are used in practice. For example, the Affordable Medicines Facility-malaria (AMFm) for ACTs is a purchase subsidy, which reduces the acquisition cost of these medicines for the distribution channel (Arrow et al. 2004, Adeyi and Atun 2010). Voucher schemes have been used to implement sales subsidies for products such as insecticide-treated bednets (ITNs) used in the prevention of malaria. The voucher provides a means by which the subsidy provider can verify a retailer's sales to end consumers. A consumer presents a voucher when purchasing an ITN and receives a discount. For each redeemed voucher the retailer submits, the retailer receives a subsidy payment (Mushi et al. 2003).

The purpose of this paper is to compare the effectiveness of a purchase subsidy versus a sales subsidy from the perspective of the donor and society. We focus on this question at the point where products are made available to consumers, the retail outlet. We find that the attractiveness of the subsidy type depends crucially on whether consumers are homogeneous or heterogeneous in their valuation of the product. For the case where consumers' valuations are heterogeneous, we provide evidence that the sales subsidy leads to greater expected donor utility, consumption, and expected social welfare. When consumers' valuations are homogeneous, we establish the opposite conclusion: the purchase subsidy leads to greater expected donor utility, consumption, and expected social welfare.

The remainder of the paper is organized as follows. §2 reviews the relevant literature. §3 describes the model. §4 examines the case with homogeneous consumers and §5 the case with heterogeneous consumers. §6 provides concluding remarks.

2 Literature Review

Because this paper takes the perspective of a donor, the literature that examines the underlying motivations of donors is relevant (Arrow 1972). Schwartz (1970), Becker (1974) and Sugden (1982) argue that individuals and private charities, in making philanthropic donation decisions, are utility maximizers, where the donor's utility depends on her contributions to the utility of others. Our paper is in this spirit.

Several papers in the economics literature starting with Pigou (1932) have examined the design of subsidies to encourage consumption of products with positive externalities. In this stream of literature, the donor's utility depends on the recipient's consumption of the product (e.g., Ben-Zion and Spiegel 1983). Daly and Giertz (1972) argue that if the donor's utility depends on the recipient's consumption choices, then the donor prefers a product subsidy to a cash transfer. In a product diffusion model, Kalish and Lilien (1983) examine how a donor should vary her product subsidy over time to accelerate consumer adoption. In a field study, Dupas (2010) finds that short-term subsidies positively impact long-term adoption of ITNs. Our work differs from these papers in that they focus on the impact of subsidies on consumption levels, assuming product availability, whereas we focus on the impact of subsidies on product availability, which in turn impacts consumption.

In an epidemiological model of malaria transmission, immunity, and drug resistance, Laxminarayan et al. (2006, 2010) study the impact of reductions in the retail price of ACTs on consumption. From simulation results under plausible parameters, they conclude that donor-funded price reductions are welfare enhancing. To capture the richness of disease progression and resistance, Laxminarayan et al. (2006, 2010) abstract away from the details of the distribution channel. We complement their macro-level approach with a micro-level approach that focuses on capturing these details: demand uncertainty, supply-demand mismatch, and the impact of subsidies on stocking and pricing decisions in the distribution channel.

Our micro-level approach is part of a stream of work in operations management that looks at the impact of incentives on the behavior of firms in a supply chain. In contrast to our focus on how donors should design subsidies to increase product availability, this literature looks at how firms should design contracts to maximize their own profits. See Cachon (2003) for an excellent review. In a classical setting, a manufacturer designs contract terms for its retailer, who faces uncertain demand in a single-period selling season. Lariviere and Porteus (2001) examine how the manufacturer should set the per-unit purchase price. A reduction in this purchase price (a so-called trade deal) is analogous to our purchase subsidy. Taylor (2002), Krishnan et al. (2004), and Aydin and Porteus (2009) examine rebates, a payment the manufacturer makes to the retailer for each unit the retailer sells to end consumers, which is analogous to our sales subsidy. Distinguished by their setting of deterministic demand that occurs continuously over a time horizon, Drèze and Bell (2003) compare trade deals with scan-back rebates (payments based on retailer sales during a specified promotion period) and find that the manufacturer prefers to offer a rebate because this eliminates the retailer's stockpiling of inventory (so-called forward buying) which occurs with trade deals.

3 Model

We focus on availability at the point where products are made available to consumers, with special attention to rural areas, because here the problem of availability is most acute, in part due to the limited reach of government-run distribution systems outside of urban areas. Because traveling longer distances is difficult for consumers, the retail market is segmented geographically, often with only one retail outlet (e.g., a drug shop or general store) for the product in a given market area, which consists of one or a small number of villages (see Goodman et al. 2009 for evidence in Tanzania).

We model product availability through the decisions of a retailer that sells to end consumers. A donor (she) offers a subsidy to the retailer (he) with the intention of increasing availability. Consumer demand depends on the market state M, a random variable with distribution F, density f, mean μ, and continuous support on $[\underline{m}, \overline{m}]$, where $0 \leq \underline{m} < \overline{m} \leq \infty$. For simplicity in exposition, we suppose that $\underline{m} = 0$ and $F(0) = 0$. Assume that F is an Increasing Failure Rate (IFR) distribution. Let $\overline{F}(\cdot) \equiv 1 - F(\cdot)$. Let $\overline{F}^{-1}(x)$ denote the inverse of $\overline{F}$ for $x \in [0, 1]$, let $\overline{F}^{-1}(x) = 0$ for $x > 1$ and $\overline{F}^{-1}(x) = \overline{m}$ for $x < 0$.

The sequence of events is as follows: First, the donor offers a per-unit sales purchase subsidy $a \geq 0$ and/or a per-unit sales subsidy $s \geq 0$. The retailer acquires Q units, incurring per-unit acquisition cost $c > 0$ less the per-unit purchase subsidy a. The market state uncertainty is resolved, and the retailer observes the market state $M = m$. The retailer sets the retail price $p \geq 0$, and demand $D_k(m, p)$ is realized, where $k \in \{\mathtt{e}, \mathtt{o}\}$. Finally, for each unit the retailer sells, the retailer receives the retail price p from the end consumer, the donor pays the retailer the sales subsidy s, and the donor receives utility $v \geq 0$. We refer to v as the donor's value; a is mnemonic for the acquisition cost reduction conferred by the purchase subsidy. Instead of explicitly modeling the cost to administer each type of subsidy, we discuss the impact of such costs in §§4.3 and 5.3.

Consumption of a unit generates utility both for the immediate consumer and for the larger society, and it is this externality benefit that motivates the donor's interest in expanding the availability of the product. Let $e \geq 0$ denote the value that consumption of a unit generates for society, excluding the immediate consumer; we refer to e as the externality. Thus, if the immediate consumer values a unit of consumption at r', then the social value derived by the consumption of that unit is $e + r'$. For most donors, the externality is an upperbound on the value that a donor places on consumption of a unit $v \leq e$, although we do not impose this as a requirement of our analysis.

We consider two demand models

$$D_{\mathtt{e}}(m, p) = \begin{cases} (m-p)/b & \text{if } p \leq m \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$D_{\mathtt{o}}(m, p) = \begin{cases} m & \text{if } p \leq r \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In (1), demand decreases linearly in the retail price. This corresponds to the setting where consumers are heterogeneous in their valuation for the product, specifically, where consumer valuations are uniformly distributed on $[0, m]$ with density $1/b$. In (2), demand is of a step-function form. This corresponds to the setting where consumers share a homogeneous valuation for the product, given by r.

In practice, consumer valuations may be relatively homogeneous or heterogeneous. Three factors favor homogeneity in the consumers' valuations. First, a retail market that is small and geographically concentrated will tend to exhibit limited heterogeneity. Second, severe income constraints in some markets prevent large differentials in the willingness to pay for the product. Third, many products with positive externalities are sold alongside similar products that lack these externalities. For example, the use of the long-standing antimalarial sulfadoxine pyremethamine (SP) contributes to the rise of drug-resistant strains of malaria, while the more recently developed ACTs are much more effective in discouraging resistance. The socially-inferior product's price serves as a reference point (Kahneman et al. 1986) which conditions consumers' valuations of the socially-superior product. To the extent that consumers are ill-informed as to the benefits of the socially-superior product, or do not believe that the benefit accrues to them personally, they may be unwilling to pay more for the superior product. As a consequence, consumer valuations for the superior product may be relatively concentrated around the price of the competitor product. When none of these factors is prominent, consumers' valuations may be relatively heterogeneous. If, in addition, the benefits that the product confers depend heavily on idiosyncratic characteristics of the consumer (e.g., the severity of the consumer's illness), this heterogeneity may be pronounced.

As noted above, we are especially interested in a retailer serving a rural market, because here the problem of availability is most acute. High travel costs for shop owners, who must often travel long distances (e.g., to the capital city) to purchase the product, mean that shop owners purchase infrequently, with limited opportunity to reorder during the selling season (e.g., the rainy season for malaria medicines). Because of the long lead time from the point of manufacturer to the point of retail sale, the remaining shelf-life of a product is often sufficiently short that unsold product from one season cannot be sold in the subsequent season (or campaign). These factors support our assumption of a single ordering opportunity. Our assumption that the retailer sets his market price in response to market conditions reflects the fact that, once the season is underway, it much easier for a retailer to adjust his price than his acquisition quantity. Rural retailers in developing countries have substantial discretion in adjusting their prices in response to market conditions: typically, they do not advertise or otherwise post prices in advance, and they face little in the way of regulatory scrutiny.

The homogeneous valuation case (equation (2)) is consistent with the classical newsvendor setting in which the retail price is exogenous, which has received substantial attention in the literature (e.g., Lariviere and Porteus 2001, Krishnan et al. 2004). A separate stream of papers (e.g., Petruzzi and Dada 1999, Drèze and Bell 2003, Aydin and Porteus 2009) considers consumers that are heterogeneous in their valuations. We seek to explore both settings in a unified framework, which allows us to explore the impact of consumer heterogeneity on optimal subsidy design.

4 Homogeneous Valuations

This section examines the case where consumers' valuations are homogeneous (demand is given by (2)), which is an approximation of the case where valuation heterogeneity is relatively limited. Although our primary focus is to compare the attractiveness to the donor of the two subsidy types, for compactness, we begin by examining the retailer's problem under a purchase and sales subsidy, because this provides insight into the retailer's problem when he receives only one type of subsidy. Regardless of the subsidy the retailer receives, it is optimal for the retailer to choose retail price $p = r$. Under a purchase and sales subsidy, the retailer's problem is to choose his acquisition quantity Q to maximize his expected profit

$$\max_{Q \geq 0} \left\{ (r+s)E\min(M,Q) - (c-a)Q \right\}. \tag{3}$$

The retailer pays the acquisition cost less the purchase subsidy $c-a$ for each unit he acquires, and then receives the sum of the retail price and the sales subsidy $r+s$ for each unit he sells. The retailer faces a standard newsvendor problem, and his optimal acquisition quantity is

$$Q_b = \overline{F}^{-1}((c-a)/(r+s)), \tag{4}$$

where the subscript is pneumonic for *both* purchase and sales subsidy.

We conclude this subsection with two benchmarks. Under no subsidy ($a = s = 0$), the retailer's optimal acquisition quantity is $\underline{Q} = \overline{F}^{-1}(c/r)$. Expected social welfare under acquisition quantity Q is the sum of the utility derived by consumers, the retailer and the rest of society for each unit consumed, $e + r$, less the cost of acquiring the Q units

$$W(Q) \equiv (e+r)E\min(M,Q) - cQ. \tag{5}$$

The quantity which maximizes expected social welfare is $\overline{Q} = \overline{F}^{-1}(c/(e+r))$.

4.1 Purchase Subsidy

The retailer's acquisition quantity problem under a purchase subsidy is a special case of (3) where the sales subsidy $s = 0$. The retailer's optimal order quantity is

$$Q_a = \overline{F}^{-1}((c-a)/r). \tag{6}$$

Under a purchase subsidy, the donor's problem is to choose the per-unit purchase subsidy $a \geq 0$ to maximize her expected utility

$$vE\min(M, Q_a) - aQ_a.$$

The donor receives utility v for each unit the retailer sells, but must pay the retailer the purchase subsidy a for each unit the retailer acquires. In offering a purchase subsidy, the donor trades off the benefit of a stochastically larger sales quantity against the up-front cost of a larger subsidy payment to the retailer.

From (6), the subsidy required to induce acquisition quantity Q_a is

$$a = c - r\overline{F}(Q_a). \tag{7}$$

Because there is a one-to-one mapping between the donor's purchase subsidy and the retailer's quantity, we can rewrite the donor's problem as one of choosing the quantity $Q \in [\underline{Q}, \overline{m})$ to maximize

$$U_a(Q) \equiv vE\min(M, Q) - cQ + r\overline{F}(Q)Q. \tag{8}$$

We observe in the appendix (Lemma 2) that $U_a(Q)$ is unimodal in Q.

Proposition 1 characterizes the donor's optimal purchase subsidy a^* and the resulting retailer's acquisition quantity Q_a^*. All proofs are in the appendix.

Proposition 1 *If*

$$v > \max((r^2/c)f(\underline{Q})\underline{Q}, c - r), \tag{9}$$

then the donor's optimal purchase subsidy a^* is the unique solution to

$$cv - (r+v)a^* - r^2\overline{F}^{-1}((c-a^*)/r)f(\overline{F}^{-1}((c-a^*)/r) = 0; \tag{10}$$

$a^* > 0$*; and the optimal resulting retailer's acquisition quantity Q_a^* is given by the unique solution to*

$$v\overline{F}(Q_a^*) - c + r[\overline{F}(Q_a^*) - Q_a^* f(Q_a^*)] = 0. \tag{11}$$

Otherwise, the optimal purchase subsidy $a^* = 0$, and the resulting acquisition quantity is $Q_a^* = \underline{Q}$.

In the case with homogeneous valuations, the retailer's acquisition quantity Q_a^* determines the level of product availability. As one would expect, product availability is increasing in the donor's value v and decreasing in the retailer's acquisition cost c. The impact of the consumers' valuation r is more subtle, as Proposition 2a demonstrates. Let τ denote the unique solution to

$$\tau - \overline{F}^{-1}(\tau)f(\overline{F}^{-1}(\tau)) = 0. \tag{12}$$

Note that $\tau \in (0, 1)$. Let $W_a^* = W(Q_a^*)$ denote expected social welfare under the optimal purchase subsidy.

Proposition 2 *Suppose (9) holds. (a) Under the optimal purchase subsidy a^*, the retailer's optimal acquisition quantity Q_a^* is strictly decreasing in the consumers' valuation r if and only if*

$$c/v < \tau. \tag{13}$$

(b) The optimal purchase subsidy a^* is strictly decreasing in the consumers' valuation r and strictly increasing in the acquisition cost c and donor value v. (c) If (13) holds, then there exists finite $\overline{e}$ such that if the externality $e > \overline{e}$, then expected social welfare under the optimal purchase subsidy W_a^* is strictly decreasing in the consumers' valuation r.

The assumption in the first line of the Proposition is innocuous in that it simply restricts attention to the interesting region where the optimal purchase subsidy is non-zero.

It is natural to expect that as the consumers' valuation of a product increases (e.g., due to improved product quality), product availability and social welfare would increase. Proposition 2 reveals that the opposite may occur. When the retailer's acquisition cost is small relative to the donor's value, product availability *decreases* as the consumers' valuation of the product increases (Proposition 2a). When, in addition, the externality e is large, expected social welfare *decreases* as the consumers' valuation of the product increases (Proposition 2c). Intuitively, an increase in the consumers' valuation r (and hence the retail price p) makes it attractive for the retailer to stock more aggressively; consequently, the donor responds by reducing her subsidy a^* (Proposition 2b). When the retailer's acquisition cost is small and the donor's value is large, this subsidy-reduction effect outweighs the valuation-increase effect, and the retailer reduces his acquisition quantity Q_a^*. When the externality e is large, the negative impact of this reduced availability on social welfare outweighs the the positive impact from the higher value r generated per unit consumed.

4.2 Sales Subsidy

The retailer's acquisition quantity problem under a sales subsidy is a special case of (3) where the purchase subsidy $a = 0$. The retailer's optimal order quantity is

$$Q_s = \overline{F}^{-1}(c/(r+s)). \tag{14}$$

Under a sales subsidy, for each unit the retailer sells, the donor receives utility v less the sales subsidy s payment to the retailer. The donor's sales subsidy problem is to choose the per-unit sales subsidy $s \geq 0$ to maximizes her expected utility

$$(v - s)E \min(M, Q_s).$$

In designing a sales subsidy, the retailer trades off the benefit of a stochastically larger sales quantity against the cost of the subsidy payment, which is increasing in the realized demand.

From (14), the subsidy required to induce acquisition quantity Q_s is

$$s = c/\overline{F}(Q_s) - r. \tag{15}$$

Because there is a one-to-one mapping between the donor's purchase subsidy and the retailer's quantity, we can rewrite the donor's problem as one of choosing the quantity $Q \in [\underline{Q}, \infty)$ to maximize

$$U_s(Q) \equiv [r + v - c/\overline{F}(Q)]E \min(M, Q).$$

Proposition 3 characterizes the donor's optimal purchase subsidy s^* and the resulting retailer's acquisition quantity Q_s^*. The proof of the proposition establishes that $U_s(Q)$ is strictly concave.

Proposition 3 *If*

$$v > \max((r^3/c^2) E \min(M, \underline{Q}) f(\underline{Q}), c - r) \tag{16}$$

then the donor's optimal sales subsidy s^* *is the unique solution to*

$$c^2(v - s^*) - f(\overline{F}^{-1}(c/(r+s^*)) \left[(r+s^*)^3 \int_0^{\overline{F}^{-1}(c/(r+s^*))} m dF(m) + c(r+s^*)^2 \overline{F}^{-1}(c/(r+s^*)) \right] = 0; \tag{17}$$

$s^* > 0$*; and the optimal resulting firm's acquisition quantity* Q_s^* *is given by the unique solution to*

$$(r+v)\overline{F}(Q_s^*) - c\left(1 + \left[f(Q_s^*)/\overline{F}(Q_s^*)^2\right] E \min(M, Q_s^*)\right) = 0. \tag{18}$$

Otherwise, the optimal sales subsidy $s^* = 0$*, and the resulting acquisition quantity is* $Q_s^* = \underline{Q}$*.*

As in the case for the optimal purchase subsidy, under the optimal sales subsidy: as the acquisition cost c increases (through the range where the optimal subsidy is strictly positive), the optimal subsidy increases but availability decreases; and as the donor's value v increases, both the subsidy and availability increase. In contrast to the results for the optimal purchase subsidy (Proposition 2a and 2c), under the optimal sales subsidy, availability and expected social welfare always increase in the consumers' valuation r.

4.3 Comparison of Subsidies

In this section, we compare the two subsidy types from the perspective of the donor, society, consumers and the retailer–with an emphasis on the perspective of the donor, the designer of the subsidy. One might expect that the donor's preference between the subsidy types would depend on the problem parameters. Theorem 1 shows that this conjecture is false.

Let U_s^* (U_a^*, respectively) denote the donor's expected utility under the optimal sales (purchase, respectively) subsidy. Let W_s^* denote the donor's expected utility under the optimal sales subsidy, and recall that W_a^* denotes the analogous value under the optimal purchase subsidy. Observe that $U_k^* = U_k(Q_k^*)$ and $W_k^* = W(Q_k^*)$ for $k \in \{a, s\}$.

Theorem 1 *(a) The donor's expected utility is higher under the optimal purchase subsidy than under the optimal sales subsidy*

$$U_a^* \geq U_s^*,$$

where the inequality is strict if (9) *holds. (b) If* $v \leq e$*, then expected social welfare is higher under the optimal purchase subsidy than under the optimal sales subsidy*

$$W_a^* \geq W_s^*, \tag{19}$$

where the inequality is strict if (9) *holds. (c) The retailer's acquisition quantity under the optimal purchase subsidy is higher than under the optimal sales subsidy*

$$Q_a^* \geq Q_s^*,$$

where the inequality is strict if (9) *holds. Therefore, the retailer's expected sales quantity is higher under the optimal purchase subsidy than under the optimal sales subsidy.*

From Theorem 1a, the donor is better off offering a purchase subsidy than a sales subsidy. To build intuition, observe that in designing a subsidy, the donor is only concerned about the quantity Q the retailer purchases–as this determines the expected benefit $vE\min(M, Q)$ the donor receives–and the cost of inducing that acquisition quantity. Because the purchase subsidy influences the retailer's acquisition quantity more directly, it is more cost effective in encouraging the retailer to purchase a larger quantity. Formally, the cost of inducing any acquisition quantity $Q \geq \underline{Q}$ is higher under the sales subsidy: Under a purchase subsidy, the cost of inducing acquisition quantity $Q \geq \underline{Q}$ is the product of the per unit subsidy and the retailer's acquisition quantity

$$aQ = [c - r\overline{F}(Q)]Q,$$

where the equality follows from (7). Under a sales subsidy, the expected cost of inducing acquisition quantity $Q \geq \underline{Q}$ is the product of the per-unit subsidy and the retailer's expected sales quantity

$$sE\min(M, Q) = [c - r\overline{F}(Q)]\left(Q + \int_0^Q mdF(m)/\overline{F}(Q)\right),$$

where the equality follows from (15). Because the subsidy cost is higher under the sales subsidy $sE\min(M, Q) > aQ$, the purchase subsidy is superior. The superiority of the purchase subsidy is larger for larger quantities (the subsidy cost difference $sE\min(M, Q) - aQ$ is increasing in Q), which suggests that the superiority of the purchase subsidy to the donor, $U_a^* - U_s^*$, grows as the donor's value v increases.

From Theorem 1c, the retailer's optimal acquisition quantity (and hence, expected consumption) is higher under the optimal purchase subsidy than under the optimal sales subsidy. The intuition stems from the fact that the cost advantage the purchase subsidy has over the sales subsidy also holds in the marginal sense. Because the donor's marginal cost of increasing the retailer's acquisition quantity is lower under the purchase subsidy (i.e., $(\partial/\partial Q)aQ \leq (\partial/\partial Q)sE\min(M, Q)$), it is optimal for the donor to induce a larger acquisition quantity under the purchase subsidy (i.e., $Q_a^* \geq Q_s^*$).

From Theorem 1b, expected social welfare is higher under the optimal purchase subsidy than under the optimal sales subsidy, provided that the donor's value is less than the externality $v \leq e$. To see the intuition, observe that expected social welfare is determined by the retailer's acquisition quantity (see (5)). Although the donor's subsidy (of either type) increases product availability, availability is still lower than the socially optimal level

$$Q_k^* \leq \overline{Q} \tag{20}$$

for $k \in \{a, s\}$. Because the optimal purchase subsidy does more to improve product availability than the optimal sales subsidy, the purchase subsidy results in higher social welfare. The inadequate availability result (20) stems from the fact that, in designing the subsidy, the donor cares about getting products into the hands of consumers but is indifferent to the retailer's profitability. (For completeness, we note that in the unlikely case that the donor values consumption at a much higher rate than the externality $v >> e$, inequalities (19) and (20) can be reversed.)

We illustrate our results using ACTs as a case example. We consider the setting where consumer valuations' are anchored by the price of the socially-inferior competitor product, SP. We use the retail price of SP (ACTwatch 2008) as our estimate for the consumers' valuation r. We use data from Kindermans et al. (2007), Mosha et al. (2010) and Patouillard et al. (2010) to estimate c, data from Chima et al. (2003) to estimate v and e , and data from ACTwatch (2008) to estimate μ. We obtain $c = 1.63$, $v = 2.20$, $e = 13.83$, $r = 0.54$, and $\mu = 435$. There is lack of systematic measurement of demand variability in field studies; we simply assume that M is a Normal(μ, σ) random variable, truncated such that its probability mass is distributed over $x \geq 0$ and that $\sigma = \mu/2$. Without a subsidy, the retailer's optimal acquisition quantity $\underline{Q} = 0$. The optimal purchase subsidy $a^* = 1.25$ results in retailer acquisition quantity $Q_a^* = 329$, expected sales of 293, expected donor utility $U_a^* = 235$, and expected social welfare $W_a^* = 3680$. The optimal sales subsidy $s^* = 1.41$ results in significantly lower availability, donor utility and social welfare: $Q_s^* = 239$, expected sales of 224, $U_s^* = 177$ and $W_s^* = 2835$. This illustrates that the donor's choice of subsidy type can have a substantial impact on the donor, consumers, and society. The results are qualitatively similar when $\sigma \in \{\mu/4, 3\mu/4\}$.

In our analysis we have ignored the administrative costs associated with implementing either subsidy. Typically, one would expect these costs to be higher under a sales subsidy because of the effort required in verifying the retailer's sales to end consumers. Taking into account these administrative costs would reinforce Theorem 1's conclusion that the donor and society are better off under a purchase subsidy.

The donor would prefer to offer a purchase subsidy rather than a sales subsidy. Ignoring the administrative costs entailed with offering both subsidies, would the donor be better off offering both subsidies types simultaneously? When offering both types of subsidies, the donor's problem is to choose the per-unit purchase subsidy $a \geq 0$ and per-unit sales subsidy $s \geq 0$ to maximize her expected utility

$$(v - s)E\min(M, Q_b) - aQ_b,$$

where, recall, (4) specifies the retailer's optimal acquisition quantity Q_b. Let (a_b^*, s_b^*) denote the donor's optimal purchase and sales subsidy. Recall that a^* denotes the optimal purchase subsidy, which is characterized in Proposition 1.

Proposition 4 *The purchase and sales subsidy that maximizes the donor's expected utility has no sales subsidy. The optimal purchase and sales subsidy is simply the optimal purchase subsidy:* $(a_b^*, s_b^*) = (a^*, 0)$.

One might expect that the donor would be better off by employing both subsidies simultaneously rather than restricting herself to the simple purchase subsidy, at least for some problem parameters. Proposition 4 shows this conjecture is false. The rationale follows that of the intuition described above for Theorem 1a: The subsidy cost for inducing any acquisition quantity is lower using the purchase subsidy than the sales subsidy. Because the sales subsidy "lever" is inferior to the purchase subsidy "lever," the donor–when given the opportunity to use both levers–only chooses the purchase subsidy lever.

5 Heterogeneous Valuations

This section examines the case where consumers' valuations are heterogeneous (demand is given by (1)). As in §4, although we are focused on the comparison between the two subsidy types, for compactness we present the retailer's problem under a purchase and sales subsidy. We first examine the retailer's pricing problem and then step back to his procurement problem. After having purchased Q units, upon observing market state $M = m$, the retailer's retail price setting problem is to choose his retail price $p \in [0, \infty)$ to maximize his expected revenue

$$(s + p)\min(D_{\mathsf{e}}(m, p), Q).$$

The retailer's retail price setting problem can be written equivalently as the problem of setting the sales quantity $q \in [0, \min(Q, m/b)]$ to maximize the expected revenue

$$(s + m - bq)q.$$

The optimal sales quantity is

$$q^* = \min(Q, m/b, (s + m)/(2b)), \tag{21}$$

which corresponds to an optimal retail price

$$p^* = m - \min(bQ, m, (s+m)/2). \tag{22}$$

Under a purchase and sales subsidy, the retailer's procurement problem is to choose his acquisition quantity $Q \geq 0$ to maximize his expected profit

$$\pi(Q) \equiv \begin{cases} \int_0^{bQ} sm/b \; dF(m) + \int_{bQ}^{\overline{m}} (s+m-bQ)Q \; dF(m) - (c-a)Q & \text{if } Q < s/b \\ \int_0^{s} sm/b \; dF(m) + \int_{s}^{2bQ-s} (s+m)^2/(4b) \; dF(m) & \\ \qquad + \int_{2bQ-s}^{\overline{m}} (s+m-bQ)Q \; dF(m) - (c-a)Q & \text{otherwise.} \end{cases}$$

Let $z(a)$ denote the unique solution to

$$\int_z^{\overline{m}} (m-z)dF(m) = c - a \tag{23}$$

if $a + \mu - c > 0$, and let $z(a) = 0$ otherwise. We abuse notation by letting $\underline{Q}$, $\overline{Q}$, Q_b and Q_k^* for $k \in \{a, s\}$ have the same meaning in the heterogeneous consumer valuation case as they had in the homogeneous valuation case. For example, Q_b denotes the retailer's optimal acquisition quantity under a purchase and sales subsidy.

Lemma 1 *Under a purchase and sales subsidy, the retailer's optimal acquisition quantity Q_b is given by the following. If $a + s + \mu - c \leq 0$, then $Q_b = 0$. Suppose instead that $a + s + \mu - c > 0$. If $s \leq z(a)$, then $Q_b = [s + z(a)]/(2b)$; further, $Q_b \geq s/b$. If $s > z(a)$, then Q_b is the unique solution to*

$$\int_{bQ_b}^{\overline{m}} (m + s - 2bQ_b)dF(m) = c - a; \tag{24}$$

further, $Q_b < s/b$.

We conclude this subsection by discussing two benchmarks: no subsidies and social welfare maximization. Under no subsidies ($a = s = 0$), the retailer's optimal acquisition quantity is $\underline{Q} = z(0)/(2b)$. Expected social welfare under acquisition quantity Q is the sum of the utility derived by consumers, the retailer and the rest of society for each unit consumed less the cost of acquiring the Q units. For any acquisition quantity Q and any realized market condition $M = m$, social welfare is maximized by selling $\min(Q, m/b)$ units to the $\min(Q, m/b)$ consumers with the highest valuations. Consequently, expected social welfare under acquisition quantity Q and the social welfare maximizing sales quantity is

$$w(Q) \equiv \int_0^{bQ} m^2/(2b)dF(m) + \int_{bQ}^{\overline{m}} (m - bQ/2)QdF(m) + eE\min(M/b, Q) - cQ.$$

The quantity which maximizes expected social welfare, $\overline{Q}$, is the unique solution to

$$\int_{b\overline{Q}}^{\overline{m}} (e + m - b\overline{Q})dF(m) = c$$

if $e + \mu - c > 0$; otherwise, $\overline{Q} = 0$. Thus, when the pricing and acquisition quantity decisions are made to maximize expected social welfare, expected social welfare is $w(\overline{Q})$.

5.1 Purchase Subsidy

Under a purchase subsidy, the retailer's procurement problem is to choose his acquisition quantity Q to maximize his expected profit $\pi(Q)|_{s=0}$. The retailer's optimal acquisition quantity under a purchase subsidy is given by the special case of Lemma 1 where $s = 0$. Under a purchase subsidy, the donor's problem is to choose the per-unit purchase subsidy $a \geq 0$ to maximize her expected utility

$$u_a \equiv vE \min(M/(2b), Q) - aQ,$$

where the retailer's optimal acquisition quantity $Q = z(a)/(2b)$. Offering a more generous purchase subsidy (increasing a) has two effects. First, it increases the retailer's acquisition quantity: $z(a)/(2b)$ is increasing in a. This first effect has a second effect: having a larger quantity encourages the retailer to price more aggressively (sell a larger quantity) when market conditions are strong. To be concrete, if a donor increases his purchase subsidy from a' to some higher level, the retailer will respond by setting a lower price when the market condition is strong $m > z(a')$. The retailer's pricing decision will be unaffected when the market condition is weak $m \leq z(a')$. In designing a purchase subsidy, the donor trades off the up-front cost of paying out the subsidy against the benefit, which only occurs when market conditions are strong, of the retailer selling a larger quantity.

Proposition 5 characterizes the donor's optimal purchase subsidy a^* and the resulting retailer's acquisition quantity Q_a^*.

Proposition 5 *(a) If* $c \in \left[0, \int_v^{\overline{m}} (m - v)dF(m)\right]$, *then the optimal purchase subsidy* $a^* = 0$ *and the retailer's optimal acquisition quantity* $Q_a^* = \underline{Q} \geq v/(2b)$. *If* $c \in \left(\int_v^{\overline{m}} (m - v)dF(m), \mu + v\right)$, *then* Q_a^* *is the unique solution to*

$$\int_{2bQ_a^*}^{\overline{m}} (v + m - 4bQ_a^*)dF(m) = c; \tag{25}$$

$Q_a^* \in (\underline{Q}, v/(2b))$; *the optimal purchase subsidy is*

$$a^* = (v - 2bQ_a^*)\overline{F}(2bQ_a^*); \tag{26}$$

and $a^* > 0$. *If* $c \geq \mu + v$, *then* $a^* = 0$ *and* $Q_a^* = 0$. *(b) The optimal purchase subsidy* a^* *is strictly* *increasing in the retailer's acquisition cost on* $c \in \left(\int_v^{\overline{m}} (m-v)dF(m), \mu+v\right)$ *with* $\lim_{c\to(\mu+v)^-} a^* = v$.

Proposition 5 provides insight into how the retailer's acquisition cost c impacts the donor's optimal purchase subsidy a^*. When the acquisition cost is low $c \in \left[0, \int_v^m (m-v)dF(M)\right]$, the retailer purchases a large quantity without a subsidy and a strictly positive per-unit subsidy is costly to the donor because it applies to retailer's large acquisition quantity; consequently, the donor offers no subsidy ($a^* = 0$). As the retailer's acquisition cost increases through the range $c \in \left(\int_v^{\overline{m}} (m-v)dF(m), \mu+v\right)$, this increasing cost discourages the retailer from purchasing units, and the donor responds by increasing the subsidy she offers. However, when the retailer's acquisition cost crosses the threshold $c = \mu + v$, the donor goes from offering a very generous subsidy, $\lim_{c\to(\mu+v)^-} a^* = v$, to offering no subsidy, $\lim_{c\to(\mu+v)+} a^* = 0$. At this cost threshold, it becomes too costly for the donor to offer a subsidy that will impact the retailer's acquisition decision, so the donor "gives up" and offers no subsidy. (The result for the case with homogenous valuations is similar; see Proposition 2b.)

Expected social welfare under the optimal purchase subsidy is the sum of the utility captured by the retailer and consumers plus the value that consumption generates for the rest of society, less the cost of acquiring the units

$$w_a^* \equiv \int_0^{2bQ_a^*} 3m^2/(8b)dF(m) + \int_{2bQ_a^*}^{\overline{m}} (m - bQ_a^*/2)Q_a^* dF(m) + eE\min(M/(2b), Q_a^*) - cQ_a^*.$$

The utility generated by consumption for the retailer, consumers and the rest of society depends on both how aggressively the retailer prices ex post, which in turn depends on how many units the retailer acquired ex ante. Relative to the social welfare maximizing decisions, under the optimal purchase subsidy, the retailer acquires too few units ex ante (provided that the donor's value is less than the externality $v \le e$), and, given this quantity, does not price aggressively enough ex post.

5.2 Sales Subsidy

Under a purchase subsidy, the retailer's procurement problem is to choose his acquisition quantity Q to maximize his expected profit $\pi(Q)|_{a=0}$. The retailer's optimal acquisition quantity under a sales subsidy is given by the special case of Lemma 1 where $a = 0$. To ease notation, for the remainder of the paper, we write $z(0)$ as simply z. Under a sales subsidy, the donor's problem is to choose the per-unit sales subsidy $s \ge 0$ to maximize her expected utility

$$u_s \equiv \begin{cases} (v-s)\left(\int_0^s m/b\ dF(m) + \int_s^{2bQ-s} (s+m)/(2b)\ dF(m) + Q\overline{F}(2bQ-s)\right) & \text{if } s < z \\ (v-s)E\min(M/b, Q) & \text{otherwise,} \end{cases}$$

where $Q = (s+z)/(2b)$ if $s < z$, and where Q is the unique solution to $\int_{bQ}^{\overline{m}} (m + s - 2bQ)dF(m) = c$ otherwise.

The sales subsidy is similar to the purchase subsidy in that both encourage the retailer to purchase a larger quantity and to price more aggressively when market conditions are strong. The subsidies differ in the degree to which they encourage more aggressive pricing. Recall that under a purchase subsidy, the retailer's more aggressive pricing was solely due to the retailer's having purchased a larger quantity. Under a sales subsidy, there is a second effect that encourages the retailer to price even more aggressively. Under a sales subsidy, the retailer receives not only the retail price but also the sales subsidy for each unit he sells, which makes it attractive for the retailer to cut his price further so as to increase the volume of units that are eligible for the subsidy. Consequently, for any acquisition quantity Q and any realized market condition $M = m$, the retailer's optimal retail price (see (22)) is lower under the sales subsidy than under the purchase subsidy, strictly so if and only if the market condition is weak $m < \max(bQ, 2bQ - s)$.

To summarize, offering a more generous sales subsidy stochastically increases the retailer's sales quantity not only by encouraging the retailer to purchase more ex ante, but also by rewarding the retailer for his sales volume ex post. In designing a sales subsidy, the donor trades off this benefit against the cost of the subsidy payment, which increases in the retailer's sales quantity.

Proposition 6 characterizes the donor's optimal sales subsidy s^* and the resulting retailer's acquisition quantity Q_s^*. Note $\mu - (v - z)\overline{F}(z)/2 < \mu + v$ if and only if the donor's value is sufficiently large $v > z\overline{F}(z)/[3 - F(z)]$.

Proposition 6 *If $c \in [0, \mu - v]$, then the optimal sales subsidy $s^* = 0$ and the retailer's optimal acquisition quantity $Q_s^* = \underline{Q}$. If $c \in \left(\mu - v, \min(\mu - (v - z)\overline{F}(z)/2, \mu + v)\right)$, then s^* is the unique solution to*

$$(v - 2s^*)\overline{F}(s^*) - \int_0^{s^*} mdF(m) - \mu + c = 0; \tag{27}$$

the retailer's optimal acquisition quantity is

$$Q_s^* = (s^* + z)/(2b);$$

further, $s^* < bQ_s^* < z$. If $c \in [\mu - (v - z)\overline{F}(z)/2, \mu + v)$, then Q_s^* is the unique solution to

$$\begin{aligned} &\int_{bQ_s^*}^{\overline{m}} (m + v - 2bQ_s^*)dF(m) - c - E\min(M, bQ_s^*) \\ &\qquad \times \left[2 + \left(c - \int_{bQ_s^*}^{\overline{m}} (m - bQ_s^*)dF(m)\right) f(bQ_s^*)/\overline{F}(bQ_s^*)^2\right] = 0; \end{aligned} \tag{28}$$

the optimal purchase subsidy is

$$s^* = 2bQ_s^* + \left(c - \int_{bQ_s^*}^{\overline{m}} m dF(m)\right) \Big/ \overline{F}(bQ_s^*); \tag{29}$$

further, $s^* \geq bQ_s^* \geq z$*. If* $c \geq \mu + v$*, then* $s^* = 0$ *and* $Q_s^* = 0$*.*

The result for the optimal sales subsidy parallels that for the optimal purchase subsidy: The donor offers a non-zero subsidy $s^* > 0$ only if the retailer's acquisition cost is moderate.

Expected social welfare under the optimal sales subsidy is the sum of the utility captured by the retailer and consumers plus the value that consumption generates for the rest of society, less the cost of acquiring the units

$$w_s^* \equiv \begin{cases} \int_0^{2bQ_s^*-z} m^2/(2b)dF(m) + \int_{2bQ_s^*-z}^{z} (3m - 2bQ_s^* + z)(m + 2bQ_s^* - z)/(8b)dF(m) \\ \quad + \int_z^{\overline{m}} (m - bQ_s^*/2)Q_s^* dF(m) + e\left[E\min(M, 2bQ_s^* - z) + \mu - c\right]/(2b) - cQ_s^* & \text{if } Q_s^* < z/b \\ \int_0^{bQ_s^*} m^2/(2b)dF(m) + \int_{bQ_s^*}^{\overline{m}} (m - bQ_s^*/2)Q_s^* \, dF(m) + eE\min(M/b, Q_s^*) - cQ_s^* & \text{otherwise.} \end{cases}$$

Recall that under the optimal purchase subsidy, relative to the social welfare maximizing decision, the retailer does not price aggressively enough. The sales subsidy mitigates this problem by encouraging the retailer to price more aggressively. Indeed, when the acquisition cost $c \in [\mu - (v-z)\overline{F}(z)/2, \mu + v)$ so that $Q_s^* \geq z/b$, the distortion in the retail pricing decision is completely eliminated: having purchased Q units, upon observing market condition $M = m$ the retailer optimally prices so as to sell the social welfare maximizing quantity, $\min(m/b, Q)$. In this case, the only source of social welfare loss is the distortion in the acquisition quantity.

We compare expected social welfare, as well as donor utility, under the two subsidy types in the next subsection.

5.3 Comparison of Subsidies

Let u_a^* (u_s^*, respectively) denote the donor's expected utility under the optimal purchase (sales, respectively) subsidy.

Theorem 2 *(a) There exists* $\overline{v} \in [z, \infty)$ *such that if* $v \leq z$ *or* $v > \overline{v}$*, then the donor's expected utility is higher under the optimal sales subsidy than under the optimal purchase subsidy*

$$u_s^* \geq u_a^*, \tag{30}$$

where the inequality is strict if $v \in (\mu - c, z]$ *or* $v > \overline{v}$*. (b) Suppose* $e \in ((c\overline{m} - E[M^2]/4)/\mu, \infty)$*. There exists* $\overline{v} \in [z, \infty)$ *and* $\underline{v} \in (\mu - c, z]$ *such that if* $v \leq \underline{v}$ *or* $v > \overline{v}$*, then expected social welfare is* *higher under the optimal sales subsidy than under the optimal purchase subsidy*

$$w_s^* \geq w_a^*, \tag{31}$$

where the inequality is strict if $v \in (\mu - c, \underline{v})$ *or* $v > \overline{v}$.

The sales subsidy is superior to the purchase subsidy, from the perspective both of the donor and of society, provided that the donor's value v is sufficiently high or low. More precisely, if the donor's value is sufficiently large ($v > \overline{v}$) or the donor's value and the production cost are low ($c < \mu$ and $v \in (\mu - c, \underline{v}]$), then both the donor and society are strictly better off under the optimal sales subsidy rather than the optimal purchase subsidy. The social welfare result requires the caveat that the externality e be sufficiently large.

To assess whether the Theorem 2's central conclusion–that the sales subsidy is superior to the purchase subsidy, from the perspective both of the donor and society–continues to hold when the Theorem's sufficient conditions are violated, we conducted a large scale numerical study. We considered the 1452 combinations of the following parameters: $e \in \{0.25,\ 0.75,\ 1.25,\ 1.75,\ 2.25,\ 2.75,\ 3.25,\ 3.75,\ 4.25,\ 4.75,\ 5.25\}$, $v \in \{e/4,\ e/2,\ 3e/4,\ e\}$, $c \in \{0.05,\ 0.45,\ 0.85,\ 1.25,\ 1.65,\ 2.05,\ 2.45,\ 2.85,\ 3.25,\ 3.65,\ 4.05\}$, $b = 0.001$, $\mu = 1$, and $\sigma \in \{\mu/4,\ \mu/2,\ 3\mu/4\}$, where, as before, M is a truncated Normal(μ, σ) random variable. In every instance, the donor's expected utility is higher under the sales subsidy, $u_s^* \geq u_a^*$, and expected social welfare is higher under the sales subsidy, $w_s^* \geq w_a^*$. Further, in all but 1.4% of the instances, the retailer's optimal acquisition quantity is higher under the sales subsidy $Q_s^* \geq Q_a^*$; in every instance, the retailer's expected sales quantity is higher under the sales subsidy. This suggests that the conclusions of Theorem 1 (expected donor utility, expected social welfare and expected sales are higher under the purchase subsidy), which apply when consumers' valuations are homogeneous, are reversed when consumers' valuations are heterogeneous.

When consumers' valuations are homogeneous, the purchase subsidy is superior because it is more effective in influencing the retailer's key decision, his acquisition quantity (see the discussion following Theorem 1). The retailer's pricing decision is dictated by the consumers' valuation, and it is not influenced by the donor's subsidy. When consumers' valuations are heterogeneous, the donor is interested in influencing not only the retailer's purchase decision, but also the retailer's pricing decision. The sales subsidy is superior to the purchase subsidy because it provides stronger incentives for the retailer to price aggressively: When the retailer makes her pricing decision, the purchase subsidy is sunk and so has a more limited impact on the retailer's pricing decision; in contrast, the sales subsidy actively rewards the retailer for increasing her sales volume. This more aggressive pricing benefits not only the donor, but society as well.

Again we use the case of ACTs to illustrate our results. In some ACTs settings, it may be more accurate to model consumers' valuations as being heterogeneous instead of homogeneous. Per the discussion in §3, this will tend to be true when the market served by a retailer is relatively large, diverse, and lacks the strong presence of socially-inferior competitor product, such as SP. For the heterogeneous case, we assume that M is a Uniform(α, β) random variable. Based on Cohen et al. (2010), we obtain $\alpha = 0$, and $\beta = 4.9$, and $b = 0.0096$. We use the same cost and valuation parameters from §4.3. The optimal purchase subsidy $a^* = 0.466$ results in retailer acquisition quantity $Q_a^* = 79.3$, expected sales of 67.0, expected donor utility $U_a^* = 110$, and expected social welfare $W_a^* = 961$. The optimal sales subsidy $s^* = 0.610$ results in higher expected sales, donor utility and social welfare: $Q_s^* = 78.8$, expected sales of 72.5, $U_s^* = 115$ and $W_s^* = 1040$. Although the donor's utility is higher under the sales subsidy, the difference is not overwhelmingly large; it is possible that the difference could be offset by the higher administrative costs required by the sales subsidy. (Although these costs may be higher under a sales subsidy, evidence from the use of sales subsidies with ITNs indicates that the administrative costs are not prohibitive (Mulligan et al. 2008). In Tanzania, ITN subsidy administrators are testing mobile phones and text messaging as a mechanism for handling vouchers, which may further reduce these costs.)

When consumers have homogeneous valuations and when the donor has the opportunity to offer both subsidy types simultaneously, she never exercises this right: It is optimal to offer only a purchase subsidy and no sales subsidy: $a_b^* \geq 0$ and $s_b^* = 0$ (Proposition 4). This result breaks when consumers have heterogeneous valuations. Specifically, when the donor's value is large, it is optimal to offer both subsidies $a_b^* > 0$ and $s_b^* > 0$. When consumers have homogeneous valuations, in offering a subsidy, the donor is influencing one decision, the retailer's acquisition quantity. Because the purchase subsidy is strictly superior at influencing this decision (see the discussion before and after Proposition 4), it is optimal to only offer a purchase subsidy. In contrast, when consumers have heterogeneous valuations, the donor is influencing not only the retailer's acquisition quantity decision, but also her pricing decision. Because the two subsidy types impact each of these decisions differently, the donor benefits by using both subsidy types.

6 Discussion

This paper provides guidance to donors designing subsidies to improve the availability of products in the private-sector distribution channel. We show that the donor's choice of the kind of subsidy to employ depends crucially on whether consumers are homogeneous or heterogeneous in their valuation of the product. For the case where consumers' valuations are heterogeneous, we provide evidence that a sales subsidy leads to greater expected donor utility, consumption, and social welfare. When consumers' valuations are homogeneous, we establish the opposite conclusion: the purchase subsidy leads to greater expected donor utility, consumption, and social welfare. Intuitively, the purchase subsidy is more effective when the key decision in the distribution channel is the purchasing decision, as it is in the homogeneous case. The sales subsidy is more effective when the pricing (sales quantity) decision is a critical decision, as it is in the heterogeneous case.

We now discuss how our modeling assumptions impact how this conclusion should be interpreted. First, differential administrative costs to implement each type of subsidy, which we have not explicitly modeled, should be taken into account. Second, our model only captures the cases of perfectly homogeneous consumers and heterogeneous consumers whose valuations are uniformly distributed. The sharply contrasting results for these two settings suggests that, in general, the donor's choice of subsidy type may depend on the degree of heterogeneity among consumers: if consumers are relatively homogeneous, the purchase subsidy is likely to be superior; if consumers are quite heterogeneous, the sales subsidy may be superior.

Third, in our model of the donor's subsidy design problem, product availability is captured by the stocking and pricing decisions of a single firm selling to end consumers. This setup informs the donor's subsidy design decision in three settings. First, the setup is appropriate when the donor is able to offer distinct subsidies to different retailers. Second, the setup informs the donor's decision when a particular type of retailer (e.g., a drug shop) is the primary means by which consumers access the product in the region where the subsidy is offered, and retailers of this type are relatively homogeneous. Third, the setup informs the donor's decision when the distribution channel serving a particular geographic area exhibits considerable vertical integration, which can be the case in developing countries where weak contractual-enforcement mechanisms favor this structure (Jaffe and Yi 2007). In such cases, the firm in the model is interpreted as being the entire distribution channel.

References

ACTwatch. 2008. Outlet Survey Report (Baseline) Federal Republic of Nigeria 12/08.

Arrow, K. J. 1972. Gifts and exchanges. *Philosophy Public Affairs* **1**(4) 343-362.

Arrow, K. J., C. Panosian, H. Gelband. 2004. *Saving lives, buying time: Economics of malaria drugs in an age of resistance.* National Academies Press, Washington, DC.

Adeyi, O., R. Atun. 2010. Universal access to malaria medicines: Innovation in financing and delivery. *Lancet* **376**(9755) 1869-1871.

Aydin, G., E. L. Porteus. 2009. Manufacturer-to-retailer versus manufacturer-to-consumer rebates in a supply chain. N. Agrawal and S. Smith, eds. *Retail Supply Chain Management.* Springer, New York, pp. 237-270.

Becker, G. S. 1974. A theory of social interactions. *J. Pol. Econ.* **82**(6) 1063–1093.

Ben-Zion, U., U. Spiegel. 1983. Philanthropic motives and contribution policy. *Public Choice* **40**(2) 117-133.

International Finance Corporation. 2007. The business of health in Africa: Partnering with the private sector to improve people's lives.

Bustreo, F. A. Harding, H. Axelsson. 2003. Can developing countries achieve adequate improvements in child health outcomes without engaging the private sector? *Bull. World Health Organization.* **81**(12) 886-894.

Cachon, G. P. 2003. Supply chain coordination with contracts. S. Graves and T. De Kok, eds. *Handbooks in Operations Research and Management Science: Supply Chain Management.* North Holland, Amsterdam, pp. 229-339.

Cameron, A. M. Ewen, D. Ross-Degnan, D. Ball, R. Laing. 2009. Medicine prices, availability, and affordability in 36 developing and middle-income countries: A secondary analysis. *Lancet* **373**(9659) 240 - 249.

Chima, R., C. Goodman, A. Mills. 2003. The economic impact of malaria in Africa: A critical review of the evidence. *Health Policy* **63**(1) 17-36.

Cohen, J., P. Dupas, S. Schaner. 2010. Prices, diagnostic tests and the demand for malaria treatment: Evidence from a randomized trial. Working Paper, Harvard University.

Daly, G., F. Giertz. 1972. Welfare economics and welfare reform. *Amer. Econ. Rev.* **62**(1) 131-138.

Drèze, X., D. R. Bell. 2003. Creating win-win trade promotions: Theory and empirical analysis of scan-back trade deals," *Marketing Sci.* 22(1) 16-39.

Dupas, P. 2010. Short-run subsidies and long-run adoption of new health products: Evidence from a field experiment. Working Paper, UCLA.

Goodman, C., S. P. Kachur, S. Abdulla, P. Bloland, A. Mills. 2009. Concentration and drug prices in the retail market for malaria treatment in rural Tanzania. *Health Econ.* **18**(6) 727-42.

Jaffe, E. D., L. Yi. 2007. What are the drivers of channel length? Distribution reform in the People's Republic of China. *Int. Bus. Rev.* **16** 474-493.

Kahneman, D. J. L. Knetsch, R. Thaler. 1986. Fairness as a constraint on profit seeking: Entitlements in the market. *Amer. Econ. Rev.* **76**(4) 728-741.

Kalish. S, G. L. Lilien. 1983. Optimal price subsidy policy for accelerating the diffusion of innovation. *Marketing Sci.* **2**(4) 407-420.

Kindermans, J.-M., J. Pilloy, P. Olliaro, M. Gomes. 2007. Ensuring sustained ACT production and reliable artemisinin supply. *Malaria J.* **6**(125)

Krishnan, H., R. Kapuscinski, D. Butz. 2004. Coordinating contracts for decentralized supply chains with retailer promotional effort. *Management Sci.* **50**(1) 48-63.

Lariviere, M.A., E. Porteus. 2001. Selling to the newsvendor: An analysis of price-only contracts. *Manufacturing Service Oper. Management* **3**(4) 293-305.

Laxminarayan, R., M. Over, D. L. Smith. 2006. Will a global subsidy of new antimalarials delay the emergence of resistance and save lives? *Health Affairs* **25**(2) 325-336.

Laxminarayan, R., I. W. H. Parry, D. L. Smith, E. Y. Klein. 2010. Should new antimalarial drugs be subsidized? *J. Health Econ.* **29**(3) 445-456.

Mosha J. F., L. Conteh, F. Tediosi, S. Gesase, J. Bruce, D. Chandramohan, R. Gosling. 2010. Cost implications of improving malaria diagnosis: Findings from north-eastern Tanzania. *PLoS ONE* **5**(1).

Mulligan, J., J. Yukich, K. Hanson. 2008. Costs and effects of the Tanzanian national voucher scheme for insecticide-treated nets. *Malaria J.* **7**(32)

Mushi A.K., J. Armstrong-Schellenberg, H. Mponda, C. Lengeler. 2003. Targeted subsidy for malaria control with treated nets using a discount voucher system in Tanzania. *Health Policy Planning* **18**(2) 163–71.

Patouillard, E., K. G. Hanson, C. A. Goodman. 2010. Retail sector distribution chains for malaria treatment in the developing world: A review of the literature. *Malaria J.* **9**(50)

Petruzzi, N. C., M. Dada. 1999. Pricing and the newsvendor problem: A review with extensions. *Operations Res.* **47**(2) 183-194.

Pigou, A. C. 1932. *The Economics of Welfare.* Macmillan, London.

Perrin, R. K., G. M. Scobie. 1981 Market intervention policies for increasing the consumption of nutrients by low income households. *Amer. J. Agr. Econ.* **63**(1) 73-82.

Sugden, R. 1982 On the economics of philanthropy. *Econ. J.* **92**(366) 341-350

Sabot O., S. Yeung, F. Pagnoni, M.Gordon, N. Petty, K. Schmits, A. Talisuna. 2008. Distribution of artemisinin-based combination therapies through private-sector channels. Resources for the Future Discussion Paper 08-43.

Saulo, E. C., B. C. Forsberg, Z. Premji, S. M. Montgomery, A. Björkman. 2008. Willingness and ability to pay for artemisinin-based combination therapy in rural Tanzania. *Malaria J.* **7**(227)

Schwartz, R. A. 1970. Personal philanthropic contributions. *J. Pol. Econ.* **78**(6) 1264-1291.

Taylor, T. A. 2002. Supply chain coordination under channel rebates with sales effort effects. *Management Sci.* **48**(8) 992-1007.

Yadav, P., O. Stapleton, L. N. Van Wassenhove. 2010. If access is the symptom, what is the cause? Comparing medicine and consumer product supply chains in the developing world. Working Paper, INSEAD.

Appendix

Lemma 2 is useful in the proofs of Propositions 1 and 2. Let

$$\Psi(Q) = v - \frac{c}{\overline{F}(Q)} + r\left[1 - \frac{Qf(Q)}{\overline{F}(Q)}\right]. \tag{32}$$

Because F is IFR, $\Psi(\cdot)$ is strictly decreasing. If $v \leq c - r$, then let $Q_a^o = 0$. If $v > c - r$, then let Q_a^o denote the unique solution to $\Psi(Q_a^o) = 0$; because $\Psi(0) > 0$ and $\Psi(\overline{m}) < 0$, $Q_a^o \in (0, \overline{m})$.It is useful to observe that

$$(\partial/\partial Q)U_a(Q) = v\overline{F}(Q) - c + r[\overline{F}(Q) - Qf(Q)].$$

Lemma 2 *The donor's expected utility under a purchase subsidy $U_a(Q)$ is unimodal on $Q \in [0, \overline{m})$, strictly increasing on $Q \in [0, Q_a^o)$, strictly decreasing on $Q \in (Q_a^o, \overline{m})$, and strictly concave on $Q \in [0, Q_a^o]$.*

The proof is similar to that of Theorem 1 in Lariviere and Porteus (2001).
Proof of Lemma 2: Note that

$$(\partial/\partial Q)U_a(Q) = \overline{F}(Q)\Psi(Q).$$

On $Q \in (Q_a^o, \overline{m})$, $\overline{F}(Q) > 0$ and $\Psi(Q) < 0$, which implies $(\partial/\partial Q)U_a(Q) < 0$. If $v \leq c - r$, then $Q_a^o = 0$ and the proof is complete. Suppose instead that $v > c - r$. Because F is IFR, $\Psi(Q)$ is strictly decreasing; similarly, $\overline{F}(Q)$ is strictly decreasing. On $Q \in [0, Q_a^o]$, $\overline{F}(Q) > 0$ and $\Psi(Q) \geq 0$, where the inequality is strict if $Q \in [0, Q_a^o)$. Therefore, $(\partial/\partial Q)U_a(Q) > 0$ on $Q \in [0, Q_a^o)$ and $(\partial^2/\partial Q^2)U_a(Q) < 0$ on $Q \in [0, Q_a^o]$.■
Proof of Proposition 1: From Lemma 2, if $v \leq c - r$, then $U_a(Q)$ is strictly decreasing on $Q \in (0, \overline{m})$. The quantity $Q \in [\underline{Q}, \overline{m})$ which maximizes the donor's expected utility $U_a(Q)$ is $Q_a^* = \underline{Q}$, which corresponds to a purchase subsidy of $a^* = 0$.

For the remainder of the proof suppose that $v > c - r$. Then, from Lemma 2, $U_a(Q)$ is unimodal with an interior maximizer. Therefore, there exists a unique solution $Q \in (0, \overline{m})$ to

$$v\overline{F}(Q) - c + r[\overline{F}(Q) - Qf(Q)] = 0. \tag{33}$$

Observe that

$$v > (r^2/c)f(\underline{Q})\underline{Q} \tag{34}$$

holds if and only if $(\partial/\partial Q)U_a(\underline{Q}) > 0$. Therefore, if (34) is violated, then $(\partial/\partial Q)U_a(Q) < 0$ on $Q \in [\underline{Q}, \overline{m})$ and the quantity $Q \in [\underline{Q}, \overline{m})$ which maximizes the donor's expected utility $U_a(Q)$ is $Q_a^* = \underline{Q}$, which corresponds to a purchase subsidy of $a^* = 0$. If (34) holds, then the unique solution to (33) has $Q > \underline{Q}$ and this quantity maximizes $U_a(Q)$. That is, the quantity that maximizes the donor's expected utility Q_a^* is the unique solution to (11). The purchase subsidy a^* corresponding to the quantity Q_a^* is strictly positive because $Q_a^* > \underline{Q}$. Further, because

$$v\overline{F}(Q_a^*) - c + r[\overline{F}(Q_a^*) - Q_a^* f(Q_a^*)]\big|_{Q_a^* = \overline{F}^{-1}((c-a^*)/r)} = 0$$

if and only if (10) holds, the optimal subsidy a^* is the unique solution to (10).■

Proof of Proposition 2: (a) To see that τ exists, is unique and satisfies $\tau \in (0,1)$, consider the quantity on the left hand side of (12). With the the change of variable $T = \overline{F}^{-1}(\tau)$, this quantity becomes $\overline{F}(T) - Tf(T)$, which has the same sign as

$$1 - Tf(T)/\overline{F}(T). \tag{35}$$

Because F is IFR, (35) is strictly decreasing in T. Because at $T = 0$, (35) is strictly positive and because $\lim_{T\to\overline{m}}[1 - Tf(T)/\overline{F}(T)] < 0$, $1 - \tilde{T}f(\tilde{T})/\overline{F}(\tilde{T}) = 0$ has a unique solution and $\tilde{T} \in (0, \overline{m})$. Therefore, (12) has a unique solution τ and further $\tau \in (0,1)$. In addition, $x - \overline{F}^{-1}(x)f(\overline{F}^{-1}(x)) < 0$ if and only if $x < \tau$.

Because (9) holds, the retailer's optimal acquisition quantity Q_a^* is the unique solution to $(\partial/\partial Q)U_a(Q_a^*) = 0$ (from Proposition 1). Further, $(\partial^2/\partial Q^2)U_a(Q_a^*) < 0$ (from Lemma 2). Because $(\partial^2/\partial r\partial Q)U_a(Q_a^*) = \overline{F}(Q_a^*) - Q_a^* f(Q_a^*)$ and because $\lim_{r\to 0^+} Q_a^* = \overline{F}^{-1}(c/v)$ (from Proposition 1),

$$\lim_{r\to 0^+}(\partial^2/\partial r\partial Q)U_a(Q_a^*) = c/v - \overline{F}^{-1}(c/v)f(\overline{F}^{-1}(c/v)). \tag{36}$$

Therefore, if $c/v < \tau$, then (36) is strictly negative and so, by the implicit function theorem, $\lim_{r\to 0^+}(\partial/\partial r)Q_a^* < 0$. This, together with the facts that $\lim_{r\to 0^+} Q_a^* = \overline{F}^{-1}(c/v)$ and Q_a^* is continuous in r, implies that there exists $\delta > 0$ such that

$$Q_a^* < \overline{F}^{-1}(c/v) \text{ for } r \in (0, \delta). \tag{37}$$

We will show by contradiction that $(\partial/\partial r)Q_a^* < 0$ for $r \in [0, \infty)$. Suppose that for some $\overline{r} > 0$, $(\partial/\partial r)Q_a^*|_{r=\overline{r}} > 0$. Because $(\partial/\partial r)Q_a^*$ is continuous in r, this implies that there exists $\underline{r} \in (0, \overline{r})$ such that $(\partial/\partial r)Q_a^*|_{r=\underline{r}} = 0$, or equivalently, $\left[\overline{F}(Q_a^*) - Q_a^* f(Q_a^*)\right]\big|_{r=\underline{r}} = 0$. Because Q_a^* satisfies (11), this implies $Q_a^*|_{r=\underline{r}} = \overline{F}^{-1}(c/v)$. This implies $(\partial/\partial r)Q_a^* = 0$ and $Q_a^* = \overline{F}^{-1}(c/v)$ for $r \in [0, \infty)$, which contradicts (37). We conclude that if $c/v < \tau$, then $(\partial/\partial r)Q_a^* < 0$ for $r \in [0, \infty)$. A parallel argument establishes that if $c/v \geq \tau$, then $(\partial/\partial r)Q_a^* \geq 0$ for $r \in [0, \infty)$, where the second inequality is strict if and only if the first inequality is strict.

(b) To characterize the impact of c and r on a^*, it is useful to first develop an expression for a^* as a function of Q_a^*. From Proposition 1, Q_a^* satisfies (11) and Q_a^* and a^* satisfy

$$cv - (r + v)a^* - r^2 Q_a^* f(Q_a^*) = 0. \tag{38}$$

Solving (11) for c, plugging this expression into (38) and rearranging terms yields

$$a^* = \overline{F}(Q_a^*)[v - rQ_a^* f(Q_a^*)/\overline{F}(Q_a^*)]. \tag{39}$$

Because $(\partial^2/\partial c\partial Q)U_a(Q) < 0$ and $(\partial^2/\partial Q^2)U_a(Q_a^*) < 0$ (from Lemma 2), by the implicit function theorem, $(\partial/\partial c)Q_a^* < 0$. Because $(\partial/\partial c)Q_a^* < 0$ and F is IFR, (39) implies that $(\partial/\partial c)a^* > 0$. Because $(\partial^2/\partial v\partial Q)U_a(Q_a^*) = \overline{F}(Q_a^*) > 0$ and $(\partial^2/\partial Q^2)U_a(Q_a^*) < 0$ (from Lemma 2), by the implicit function theorem, $(\partial/\partial v)Q_a^* > 0$. Therefore, $a^* = c - r\overline{F}(Q_a^*)$ implies $(\partial/\partial v)a^* > 0$. Suppose (13) does not hold. Then $(\partial/\partial r)Q_a^* \geq 0$. Because $(\partial/\partial r)Q_a^* \geq 0$ and F is IFR, (39) implies that $(\partial/\partial r)a^* < 0$. Suppose (13) holds. Then $(\partial/\partial r)Q_a^* < 0$. Because $a^* = c - r\overline{F}(Q_a^*)$,

$$(\partial/\partial r)a^* = -\overline{F}(Q_a^*) + rf(Q_a^*)(\partial/\partial r)Q_a^*. \tag{40}$$

Then $(\partial/\partial r)Q_a^* < 0$ implies that (40) is strictly negative.

(c) Observe that

$$(\partial/\partial r)W_a^* = E\min(M, Q_a^*) + [(e+r)\overline{F}(Q_a^*) - c](\partial/\partial r)Q_a^*. \tag{41}$$

Because (13) holds, $(\partial/\partial r)Q_a^* < 0$. Let

$$\overline{e} = [c - E\min(M, Q_a^*)/(\partial/\partial r)Q_a^*]/\,\overline{F}(Q_a^*) - r.$$

Then, from (41), if $e > \overline{e}$, then $(\partial/\partial r)W_a^* < 0$.■

Proof of Proposition 3: Note that

$$(\partial/\partial Q)U_s(Q) = (r+v)\overline{F}(Q) - c\left(1 + \left[f(Q)/\overline{F}(Q)^2\right]E\min(M, Q)\right). \tag{42}$$

Because F is IFR, (42) is strictly decreasing in Q. If $v \leq c - r$, then $U_s(Q)$ is strictly decreasing on $Q \in (0, \overline{m})$. The quantity $Q \in [\underline{Q}, \overline{m})$ which maximizes the donor's expected utility $U_s(Q)$ is $Q_s^* = \underline{Q}$, which corresponds to a sales subsidy of $s^* = 0$.

For the remainder of the proof suppose that $v > c - r$. Because (42) is strictly decreasing in Q, $(\partial/\partial Q)U_s(0) > 0$ and $(\partial/\partial Q)U_s(\overline{m}) < 0$, there exists a unique solution Q to

$$(r+v)\overline{F}(Q) - c\left(1 + \left[f(Q)/\overline{F}(Q)^2\right]E\min(M, Q)\right) = 0, \tag{43}$$

and further this solution $Q \in (0, \overline{m})$. Observe that

$$v > (r^3/c^2)E\min(M, \underline{Q})f(\underline{Q}) \tag{44}$$

holds if and only if $(\partial/\partial Q)U_s(\underline{Q}) > 0$. Therefore, if (44) is violated, then $(\partial/\partial Q)U_s(Q) < 0$ on $Q \in (\underline{Q}, \overline{m})$ and the quantity $Q \in [\underline{Q}, \overline{m})$ which maximizes the donor's expected utility $U_s(Q)$ is $Q_s^* = \underline{Q}$, which corresponds to a sales subsidy of $s^* = 0$. If (44) holds, then the unique solution to (43) has $Q > \underline{Q}$ and this quantity maximizes $U_s(Q)$. That is, the quantity that maximizes the donor's expected utility Q_s^* is the unique solution to (18). The purchase subsidy s^* corresponding to the quantity Q_s^* is strictly positive because $Q_s^* > \underline{Q}$. Further, because

$$(r+v)\overline{F}(Q_s^*) - c\left(1 + \left[f(Q_s^*)/\overline{F}(Q_s^*)^2\right]E\min(M, Q_s^*)\right)\Big|_{Q_s^* = \overline{F}^{-1}(c/(r+s^*))} = 0$$

if and only if (17) holds, the optimal subsidy s^* is the unique solution to (17).■

Proof of Theorem 1: If $v \leq c - r$, then $Q_a^* = Q_s^* = \underline{Q}$ (from Propositions 1 and 3). This implies that $U_a^* = U_s^*$ and $W_a^* = W_s^*$. For the remainder of the proof suppose that $v > c - r$. The structure of proof is first to establish (c), then (b) and then (a).

(c) Because $(r^2/c)f(\underline{Q})\underline{Q} < (r^3/c^2)E\min(M,\underline{Q})f(\underline{Q})$, if $v \le (r^2/c)f(\underline{Q})\underline{Q}$ then, $Q_a^* = Q_s^* = \underline{Q}$ (from Propositions 1 and 3). If $v \in ((r^2/c)f(\underline{Q})\underline{Q}, (r^3/c^2)E\min(M,\underline{Q})f(\underline{Q}))$, then $Q_a^* > \underline{Q} = Q_s^*$. Finally, suppose that $v > (r^3/c^2)E\min(M,\underline{Q})f(\underline{Q})$. Note that

$$\begin{aligned}
(\partial/\partial Q)U_s(Q_a^*) &= (r+v)\overline{F}(Q_a^*) - c\left(1 + \left[f(Q_a^*)/\overline{F}(Q_a^*)^2\right]E\min(M,Q_a^*)\right) \\
&= f(Q_a^*)[rQ_a^* - cE\min(M,Q_a^*)/\overline{F}(Q_a^*)^2] \qquad (45) \\
&< \left[rf(Q_a^*)E\min(M,Q_a^*)/\overline{F}(Q_a^*)^2\right]\left[\overline{F}(Q_a^*) - c/r\right] \\
&< 0 \qquad (46) \\
&= (\partial/\partial Q)U_s(Q_s^*),
\end{aligned}$$

where (45) follows from (11), and (46) follows because $Q_a^* > \underline{Q}$. Because F is IFR, $(\partial/\partial Q)U_s(Q)$ is strictly decreasing in Q, which implies $Q_a^* > Q_s^*$.

(b) First we establish that

$$Q_a^* \le \overline{Q}. \qquad (47)$$

If $v \le (r^2/c)f(\underline{Q})\underline{Q}$, then $Q_a^* = \underline{Q} \le \overline{Q}$, where the equality follows from Proposition 1 and the inequality follows because $e \ge 0$. If $v > (r^2/c)f(\underline{Q})\underline{Q}$, then (47) follows from the facts that $U_a(Q)$ is unimodal in Q (from Lemma 2) and maximized at $Q = Q_a^*$ and

$$\begin{aligned}
(\partial/\partial Q)U_a(\overline{Q}) &= v\overline{F}(\overline{Q}) - c + r[\overline{F}(\overline{Q}) - \overline{Q}f(\overline{Q})] \\
&= -c(e-v)/(e+r) - r\overline{Q}f(\overline{Q}) \\
&\le 0,
\end{aligned}$$

where the inequality follows because $v \le e$. Inequalities $(r^2/c)f(\underline{Q})\underline{Q} < v$ and $v \le e$ imply $e > 0$, which implies $\underline{Q} < \overline{Q}$. Expected social welfare $W(Q)$ is strictly increasing in Q for $Q \in [\underline{Q}, \overline{Q}]$, so the result follows from Part (c), $\underline{Q} \le Q_s^*$, and (47).

(a) We will show that for $Q \in [\underline{Q}, \overline{m})$,

$$U_a(Q) \ge U_s(Q). \qquad (48)$$

Inequality (48) is equivalent to

$$0 \ge [r - c/\overline{F}(Q)]\int_0^Q m dF(m),$$

and this inequality holds because $Q \in [\underline{Q}, \overline{m})$. Then

$$U_a^* = U_a(Q_a^*) \ge U_a(Q_s^*) \ge U_s(Q_s^*) = U_s^*, \qquad (49)$$

If $v > (r^2/c)f(\underline{Q})\underline{Q}$ holds, then $Q_a^* > Q_s^*$ (from Part (c)) and because Q_a^* is unique (from Proposition 1), the first inequality in (49) is strict.■

Proof of Proposition 4: From (4), $a = c - (r+s)\overline{F}(Q_b)$. For a given sales subsidy s, there is a one-to-one mapping between the donor's purchase subsidy a and the retailer's acquisition quantity Q_b. Consequently, we can rewrite the donor's problem as one of choosing the quantity $Q \in [\underline{Q}, \overline{m})$ and sales subsidy $s \geq 0$ to maximize

$$(v-s)E\min(M,Q) - cQ + (r+s)\overline{F}(Q)Q,$$

which simplifies to

$$vE\min(M,Q) - cQ + r\overline{F}(Q)Q - s\int_0^Q m dF(m).$$

Because this quantity is strictly decreasing in s, the optimal sales subsidy is $s_b^* = 0$.■

Proof of Lemma 1: First, observe that

$$(\partial/\partial Q)\pi(Q) = \begin{cases} \int_{bQ}^{\overline{m}} (m+s-2bQ)dF(m) - c + a & \text{if } Q < s/b \\ \int_{2bQ-s}^{\overline{m}} (m+s-2bQ)dF(m) - c + a & \text{otherwise,} \end{cases}$$

and

$$(\partial^2/\partial Q^2)\pi(Q) = \begin{cases} -2b\overline{F}(bQ) - (s-bQ)bf(bQ) & \text{if } Q < s/b \\ -2b\overline{F}(2bQ-s) & \text{otherwise.} \end{cases}$$

Because $\pi(Q)$ is continuous,

$$\lim_{Q\to(s/b)^+}(\partial/\partial Q)\pi(Q) = \lim_{Q\to(s/b)^-}(\partial/\partial Q)\pi(Q) = \int_s^{\overline{m}}(m-s)dF(m) - c + a,$$

and $\pi(Q)$ is strictly concave, $\pi(Q)$ has a unique maximizer, Q_b. If $a+s+\mu-c \leq 0$, then $(\partial/\partial Q)\pi(0) \leq 0$ and so the quantity that maximizes $\pi(Q)$ is $Q_b = 0$. Suppose instead that $a+s+\mu-c > 0$. If $s \leq z(a)$, then $(\partial/\partial Q)\pi(s/b) \geq 0$ and the unique solution to the first order condition $(\partial/\partial Q)\pi(Q_b) = 0$ has quantity $Q_b = [s+z(a)]/(2b)$. If $s > z(a)$, then $(\partial/\partial Q)\pi(s/b) < 0$ and the unique solution to the first order condition $(\partial/\partial Q)\pi(Q_b) = 0$ is the unique solution to (24).■

Lemma 3 is useful in the proof of Proposition 5.

Lemma 3 *The donor's problem under a purchase subsidy* $\max_{a\geq 0} u_a$ *can be written as* $\max_{Q\in[\underline{Q},\overline{m}/(2b))} u_a(Q)$, *where*

$$u_a(Q) = \int_0^{2bQ}(vm)/(2b)\ dF(m) + \int_{2bQ}^{\overline{m}}(v+m-2bQ)QdF(m) - cQ \tag{50}$$

is the donor's expected utility as a function of the induced acquisition quantity Q that results from a given purchase subsidy.

Proof of Lemma 3: We first consider the case where $a > c-\mu$. Then, from Lemma 1, the retailer's optimal acquisition quantity Q satisfies

$$\int_{2bQ}^{\overline{m}} (m - 2bQ)dF(m) = c - a. \tag{51}$$

Because there is a one-to-one mapping between the donor's purchase subsidy and the retailer's quantity, we can use (51) to rewrite the donor's problem as one of choosing the quantity $Q \in [\underline{Q}, \overline{m}/(2b))$ to maximize $u_a(Q)$, which is specified in (50).

If $a \leq c - \mu$, then, by Lemma 1, the retailer's optimal acquisition quantity is $Q = 0$ and the donor's expected utility is $u_a = 0$. Observe that $a \leq c - \mu$ requires that $c \geq \mu$, which implies that $\underline{Q} = 0$. Thus, choosing $a \leq c - \mu$, is equivalent to choosing induced acquisition quantity $Q = \underline{Q} = 0$ which results in profit $u_a(\underline{Q}) = 0$.■

Proof of Proposition 5: (a) Because $u_a < 0$ for $a > v$ and $u_a \geq 0$ for $a \leq v$, the optimal purchase subsidy $a^* \leq v$. Therefore, if $c \geq \mu + v$, then $a^* \leq v \leq c - \mu$. From Lemma 1, because the optimal purchase subsidy $a^* \leq c - \mu$, the retailer's optimal acquisition quantity $Q_a^* = 0$ and an optimal purchase subsidy is $a^* = 0$. In the remainder of the proof, we suppose that $c < \mu + v$.

From Lemma 3, the donor's problem under a purchase subsidy can be written as $\max_{Q \in [\underline{Q}, \overline{m}/(2b))} u_a(Q)$. Note that

$$(\partial/\partial Q)u_a(Q) = \int_{2bQ}^{\overline{m}} (v + m - 4bQ)dF(m) - c.$$

We will show that

$$(\partial/\partial Q)u_a(\underline{Q}) > 0 \text{ if and only if } c > \int_v^{\overline{m}} (m - v)dF(m). \tag{52}$$

If $c \geq \mu$, then $\underline{Q} = 0$ and

$$(\partial/\partial Q)u_a(\underline{Q}) = v + \mu - c > 0. \tag{53}$$

If $c < \mu$, then $\underline{Q} = z(0)/(2b)$ and

$$\begin{aligned} (\partial/\partial Q)u_a(\underline{Q}) &= \int_{2b\underline{Q}}^{\overline{m}} (v + m - 4b\underline{Q})dF(m) - \int_{2b\underline{Q}}^{\overline{m}} (m - 2b\underline{Q})dF(m) \\ &= (v - 2b\underline{Q})\overline{F}(2b\underline{Q}), \end{aligned} \tag{54}$$

which is strictly positive if and only if $c > \int_v^{\overline{m}} (m - v)dF(m)$. This establishes (52).

Because $\lim_{Q \to \overline{m}/(2b)} (\partial/\partial Q)u_a(Q) < 0$, the retailer's optimal acquisition quantity under the optimal purchase subsidy, Q_a^*, either is the boundary solution $Q_a^* = \underline{Q}$ or satisfies the first order condition $(\partial/\partial Q)u_a(Q_a^*) = 0$, which is stated explicitly as (25). We will now show that if Q_a^* satisfies (25), then Q_a^* is the unique solution to (25). Suppose that Q_a^* satisfies (25). Then, the optimal purchase subsidy is

$$
\begin{aligned}
a^* &= c - \int_{2bQ_a^*}^{\overline{m}} (m - 2bQ_a^*) dF(m) \\
&= \int_{2bQ_a^*}^{\overline{m}} (v + m - 4bQ_a^*) dF(m) - \int_{2bQ_a^*}^{\overline{m}} (m - 2bQ_a^*) dF(m), \\
&= (v - 2bQ_a^*)\overline{F}(2bQ_a^*).
\end{aligned} \tag{55}
$$

Because $a^* \geq 0$, (55) implies

$$Q_a^* \leq v/(2b),$$

where the inequality is strict if $a^* > 0$. Further,

$$(\partial^2/\partial Q^2) u_a(Q) = -2b(v - 2bQ)f(2bQ) - 4b\overline{F}(2bQ),$$

so $u_a(Q)$ is strictly concave on $Q \in [0, v/(2b)]$, and so the retailer's optimal acquisition quantity under the optimal purchase subsidy Q_a^* is the unique solution to (25). In sum, one of the following holds: $Q_a^* = \underline{Q}$ and $a^* = 0$; or Q_a^* is the unique solution to (25), a^* is given by (26) and $a^* > 0$.

Case 1: $c \in \left[0, \int_v^{\overline{m}} (m - v) dF(m)\right]$. First observe that $c \leq \int_v^{\overline{m}} (m - v) dF(m)$ implies $\underline{Q} \geq v/(2b)$. Suppose that $a^* > 0$. Then $c \leq \mu$ implies that $a^* > c - \mu$ and $Q_a^* > \underline{Q}$. However, because Q_a^* satisfies (25), $Q_a^* \leq v/(2b)$, which contradicts $Q_a^* > \underline{Q} \geq v/(2b)$. We conclude that $a^* = 0$, which implies that $Q_a^* = \underline{Q}$.

Case 2: $c \in \left(\int_v^{\overline{m}} (m - v) dF(m), \mu + v\right)$. From (52), $(\partial/\partial Q) u_a(\underline{Q}) > 0$, which implies $Q_a^* > \underline{Q}$. Therefore, Q_a^* is the unique solution to (25), $Q_a^* \in (\underline{Q}, v/(2b))$, a^* is given by (26) and $a^* > 0$.

(b) From part (a), because $c \in \left(\int_v^{\overline{m}} (m - v) dF(m), \mu + v\right)$, Q_a^* is the unique solution to (25) and a^* is given by (26). From (25), Q_a^* is strictly decreasing in c. From (26), a^* is strictly decreasing in Q_a^*. Therefore, a^* is strictly increasing in c. Because $\lim_{c \to (\mu+v)^-} Q_a^* = 0$, (26) implies $\lim_{c \to (\mu+v)^-} a^* = v$.■

Lemma 4 is useful in the proof of Proposition 6.

Lemma 4 *The donor's problem under a purchase subsidy* $\max_{s \geq 0} u_s$ *can be written as* $\max_{Q \in [\underline{Q}, \min(\max(v,z),\overline{m})/b]} u_s(Q)$, *where*

$$u_s(Q) = \begin{cases} [(v - 2bQ + z)/(2b)] \left[E \min(M, 2bQ - z) + \mu - c\right] & \text{if } Q < z/b \\ \left[v - 2bQ - \left(c - \int_{bQ}^{\overline{m}} m dF(m)\right) \Big/ \overline{F}(bQ)\right] E \min(M/b, Q) & \text{otherwise} \end{cases} \tag{56}$$

is the donor's expected utility as a function of the induced acquisition quantity Q that results from a given sales subsidy. Further, $u_s(Q)$ is strictly concave on $Q \in [\underline{Q}, \min(\max(v,z),\overline{m})/b]$; $s < z$ if and only if the induced acquisition quantity $Q < z/b$. If the induced acquisition quantity $Q < z/b$, then the sales subsidy s that induces this acquisition quantity is

$$s = 2bQ - z; \tag{57}$$

otherwise, the sales subsidy is

$$s = 2bQ + \left(c - \int_{bQ}^{\overline{m}} m dF(m)\right) \Big/ \overline{F}(bQ). \tag{58}$$

Proof of Lemma 4: If $s \leq c - \mu$, then, by Lemma 1, the retailer's optimal acquisition quantity is $Q = 0$ and the donor's expected utility is $u_s = 0$. Observe that $s \leq c - \mu$ requires that $c \geq \mu$, which implies that $\underline{Q} = 0$. Thus, choosing $s \leq c - \mu$, is equivalent to choosing induced acquisition quantity $Q = \underline{Q} = 0$, which results in profit $u_s(\underline{Q}) = 0$.

For the remainder of the proof, suppose that $s > c - \mu$. From Lemma 1, under a sales subsidy, there is a one-to-one mapping between the donor's sales subsidy and the retailer's quantity, so we can we can write the donor's expected utility as a function of the retailer's induced acquisition quantity that results from a given sales subsidy. Because $u_s \leq 0$ for $s > v$, without loss of generality we restrict attention to $s \leq v$.

First, suppose that $s < z$. The donor's expected utility is

$$\begin{aligned} u_s &= (v - s)\left(\int_0^s \frac{m}{b} dF(m) + \int_s^z \frac{s+m}{2b} dF(m) + \frac{s+z}{2b}\overline{F}(z)\right) \\ &= [(v-s)/(2b)]\,[E\min(M,s) + E\min(M,z)] \\ &= [(v-s)/(2b)]\,[E\min(M,s) + \mu - c] \\ &= [(v - 2bQ + z)/(2b)]\,[E\min(M, 2bQ - z) + \mu - c]\,, \end{aligned} \tag{59}$$

and we denote the right hand side of (59) as $u_s(Q)$. Then

$$\begin{aligned} (\partial/\partial Q)u_s(Q) &= (v - 4bQ + 2z)\overline{F}(2bQ - z) - \int_0^{2bQ - z} m dF(m) - \mu + c \\ (\partial^2/\partial Q^2)u_s(Q) &= -2b[(v - 2bQ + z)f(2bQ - z) + 2\overline{F}(2bQ - z)] < 0, \end{aligned} \tag{60}$$

where the inequality follows from $s \leq v$ and $s = 2bQ - z$.

Second, suppose that $s \geq z$. Because the retailer's acquisition quantity Q satisfies

$$\int_{bQ}^{\overline{m}} (m + s - 2bQ) dF(m) = c, \tag{61}$$

(58) holds. Therefore, the donor's expected utility can be written as a function of the induced acquisition quantity Q as

$$u_s(Q) = \left[v - 2bQ - \left(c - \int_{bQ}^{\overline{m}} m dF(m)\right) \Big/ \overline{F}(bQ)\right] E\min(M/b, Q).$$

Then

$$(\partial/\partial Q)u_s(Q) = \mathcal{A}(Q) - \mathcal{B}(Q)[2 + \mathcal{C}(Q)\mathcal{D}(Q)], \tag{62}$$

where $\mathcal{A}(Q) = \int_{bQ}^{\overline{m}}(m + v - 2bQ)dF(m) - c$, $\mathcal{B}(Q) = E\min(M, bQ)$, $\mathcal{C}(Q) = c - \int_{bQ}^{\overline{m}}(m - bQ)dF(m)$

and $\mathcal{D}(Q) = f(bQ)/\overline{F}(bQ)^2$. We will show that

$$\begin{aligned} bQ &\geq z && (63)\\ \mathcal{C}(Q) &\geq 0. && (64) \end{aligned}$$

If $c \geq \mu$, then $z = 0$, which implies (63). Further, because $\mathcal{C}(Q) \geq c - \mu$, (64) follows. If $c < \mu$, then z satisfies (23) where $a = 0$. This, along with the fact that Q is the unique solution to (61), implies (63) and (64). Because $u_s(Q) < 0$ for $bQ > v$, without loss of generality we restrict attention to $bQ \leq v$. Note

$$\begin{aligned} (\partial^2/\partial Q^2)u_s(Q) &= (\partial/\partial Q)\mathcal{A}(Q) - (\partial/\partial Q)\mathcal{B}(Q)[2 + \mathcal{C}(Q)\mathcal{D}(Q)] \\ &\quad - \mathcal{B}(Q)[\mathcal{C}(Q)(\partial/\partial Q)\mathcal{D}(Q) + \mathcal{D}(Q)(\partial/\partial Q)C(Q). \end{aligned}$$

Further, $(\partial/\partial Q)\mathcal{A}(Q) = -b[2\overline{F}(bQ) + (v - bQ)f(bQ)] < 0$, where the inequality follows because $bQ \leq v$. Also, $(\partial/\partial Q)\mathcal{B}(Q) = (\partial/\partial Q)\mathcal{C}(Q) = b\overline{F}(bQ) > 0$. Because F is IFR, $(\partial/\partial Q)\mathcal{D}(Q) > 0$. These results collectively imply that

$$(\partial^2/\partial Q^2)u_s(Q) < 0. \tag{65}$$

Next we establish that $u_s(Q)$ is maximized on $Q \leq \overline{m}/b$. If $s \geq z$, then this follows from (62) because $(d/dQ)u_s(Q) < 0$ for $Q > \overline{m}/b$. If $s < z$, then because $(d/dQ)u_s(Q) < 0$ for $Q \geq (\overline{m}+z)/(2b)$, $u_s(Q)$ is maximized on $Q \leq (\overline{m} + z)/(2b)$, which implies $s < \overline{m}$. This, together with the fact that $z < \overline{m}$, implies that $\overline{m}/b > (s + z)/(2b) = Q$. For the remainder of the proof, for expositional simplicity, assume $\overline{m} = \infty$; adapting the proof for the $\overline{m} < \infty$ case is straightforward.

Because for $s < z$, the retailer's acquisition quantity $Q = (s + z)/(2b)$ and because for $s \geq z$, the retailer's acquisition quantity Q satisfies (61), $\lim_{s \to z^-} Q = \lim_{s \to z^+} Q = z/b$. Further,

$$s \leq z \text{ if and only if the retailer's acquisition quantity } Q \leq z/b, \tag{66}$$

where the first inequality is strict if and only if the second inequality is strict. Furthermore,

$$\begin{aligned} \lim_{Q \to (z/b)^-} (\partial/\partial Q)u_s(Q) &= \lim_{Q \to (z/b)^+} (\partial/\partial Q)u_s(Q) = (v - z)\overline{F}(z) - 2(\mu - c). && (67)\\ \lim_{Q \to (z/b)^-} u_s(Q) &= \lim_{Q \to (z/b)^+} u_s(Q) = (v - z)E\min(M, z)/b. && (68) \end{aligned}$$

Next we establish that $u_s(Q)$ is maximized on $Q \in [\underline{Q}, \max(v, z)/b]$ and is strictly concave in Q on this interval. First, suppose $v \leq z$. Because $s < v$, this implies $s \leq z$, which implies $Q \leq z/b$ (from (66)). That is, $u_s(Q)$ is maximized on $Q \in [\underline{Q}, z/b]$. Further, (60) and (67)-(68) imply that $u_s(Q)$ is strictly concave on $Q \in [\underline{Q}, z/b]$. Second, suppose $z \leq v$. As noted above, $u_s(Q)$ is maximized on $Q \in [\underline{Q}, v/b]$. Because $u_s(Q)$ is strictly concave on $Q \in [\underline{Q}, z/b)$ (from (60)) and is strictly concave on $Q \in [z/b, v/b]$ (from (65)), $u_s(Q)$ is strictly concave on $Q \in [\underline{Q}, v/b]$ (from (67)-(68)). Finally, if $Q < z/b$, then $s < z$ (from (66)); therefore (57) holds (from Lemma 1). Similarly, if $Q \geq z/b$, then $s \geq z$; therefore, (58) holds. ■

Proof of Proposition 6: From Lemma 4, the donor's problem is to choose $Q \in [\underline{Q}, \min(\max(v,z),\overline{m})/b]$ to maximize $u_s(Q)$.

Case 1: $c \in [0, \mu - v]$. Then $(\partial/\partial Q)u_s(\underline{Q}) = v - \mu + c \leq 0$. Because $u_s(Q)$ is strictly concave, this implies that $u_s(Q)$ is decreasing in Q on $Q \geq \underline{Q}$. Therefore, the optimal acquisition quantity $Q_s^* = \underline{Q}$, which corresponds to an optimal sales subsidy of $s^* = 0$.

Case 2: $c \geq \mu + v$. Now $c > \mu$ implies $z = 0$ and $\underline{Q} = 0$. Therefore, $s^* \geq z$ and $(\partial/\partial Q)u_s(\underline{Q}) = v + \mu - c \leq 0$. By the same argument in Case 1, $Q_s^* = \underline{Q}$ and $s^* = 0$.

Case 3: $c \in \left(\mu - v, \min(\mu - (v-z)\overline{F}(z)/2, \mu + v)\right)$. Then $(\partial/\partial Q)u_s(\underline{Q}) = v - \mu + c > 0$. Further, because $c < \mu - (v-z)\overline{F}(z)/2$, from (67), $(\partial/\partial Q)u_s(z/b) < 0$. So $Q_s^* < z/b$ and Q_s^* is the unique solution to the first order condition

$$(\partial/\partial Q)u_s(Q_s^*) = (v - 4bQ_s^* + 2z)\overline{F}(2bQ_s^* - z) - \int_0^{2bQ_s^*-z} m dF(m) - \mu + c = 0. \tag{69}$$

Because $Q_s^* < z/b$, the optimal sales subsidy $s^* < z$ (from Lemma 4). From Lemma 1, $Q_s^* = (s^* + z)/(2b)$. Therefore, the second equality in (69) can be rewritten as (27) and s^* is the unique solution to (27). Further, $s^* < z$ implies $s^* < bQ_s^* < z$, where the second inequality follows from Lemma 4.

Case 4: $c \in [\mu - (v-z)\overline{F}(z)/2, \mu + v)$. Because $c \geq \mu - (v-z)\overline{F}(z)/2$, from (67), $(\partial/\partial Q)u_s(z/b) \geq 0$. So $Q_s^* \geq z/b$ and Q_s^* is the unique solution to the first order condition $(\partial/\partial Q)u_s(Q_s^*) = 0$, which is equivalent to (28). Equation (29) follows from (58). It remains to show that $s^* \geq bQ_s^*$. From (29), it is sufficient to show that

$$c - \int_{bQ_s^*}^{\overline{m}} (m - bQ_s^*) dF(m) \geq 0. \tag{70}$$

If $c \geq \mu$, then (70) follows. If $c < \mu$, then z satisfies (23) where $a = 0$. This, along with the fact that $bQ_s^* \geq z$, implies (70).■

Proof of Theorem 2: (a) First, we establish the results for $v \leq z$. Suppose $c < \mu$. Then $c = \int_z^{\overline{m}}(m - z)dF(m) \leq \int_v^{\overline{m}}(m - v)dF(m)$, where the inequality holds because $v \leq z$. Thus, from Proposition 5, if $v \leq z$, then $a^* = 0$, which implies (30). From Proposition 6, if $v > \mu - c$, then $s^* > 0$ and so for $v \in (\mu - c, z]$, the inequality in (30) is strict. Suppose instead that $c \geq \mu$. Then $z = 0$. If $v = 0$, then by Propositions 5 and 6, $a^* = s^* = 0$, which implies $u_s^* = u_a^*$.

Second, we establish the results for $v > \overline{v}$. We begin by establishing a technical result that will be useful subsequently: if the donor's valuation is sufficiently large $v \geq 4z + c/\overline{F}(2z)$, then

$$Q_a^* \geq z/b. \tag{71}$$

If $c \geq \mu$, then $z = 0$, so (71) holds. Suppose instead that $c < \mu$. If $v > z$, then $c = \int_z^{\overline{m}}(m - z)dF(m) > \int_v^{\overline{m}}(m - v)dF(m)$. Therefore, from Proposition 5, Q_a^* is the unique solution to (25). If $v \geq 4z + c/\overline{F}(2z)$, then $\displaystyle\int_{2z}^{\overline{m}}(v + m - 4z)dF(m) \geq c$, and it is straightforward to show that this implies (71).

Case 1: $\overline{m} < \infty$. We will establish that (30) holds with strict inequality when $\overline{v} = \max(4z + c/\overline{F}(2z), 2[\overline{m} + c/\overline{F}(\overline{m}/2)])$. To do so, we will establish that

$$u_s(Q) > u_a(Q) \text{ for } Q \in [z/b, \min(v, \overline{m})/(2b)) \text{ and } v > 2[\overline{m} + c/\overline{F}(\overline{m}/2)]. \tag{72}$$

For $Q \in [z/b, \min(v, \overline{m})/(2b))$ and $v > 2[\overline{m} + c/\overline{F}(\overline{m}/2)]$,

$$\begin{aligned} u_s(Q) - u_a(Q) &= \frac{1}{b}\left[\int_0^{bQ} mdF(m)\left(v/2 - 2bQ - \left[c - \int_{bQ}^{\overline{m}} mdF(m)\right]\Big/\overline{F}(bQ)\right)\right. \\ &\quad \left. +(v/2 - bQ)\int_{bQ}^{2bQ}(2bQ - m)dF(m)\right] \quad \text{(because } Q \geq z/b\text{)} \qquad (73) \\ &> \frac{1}{b}\int_0^{bQ} mdF(m)[v/2 - 2bQ - c/\overline{F}(bQ)] \quad \text{(because } Q < v/2b\text{)} \\ &> \frac{1}{b}\int_0^{bQ} mdF(m)[v/2 - \overline{m} - c/\overline{F}(\overline{m}/2)] \quad \text{(because } Q < \overline{m}/2b\text{)} \\ &> 0. \qquad \text{(because } v > 2[\overline{m} + c/\overline{F}(\overline{m}/2)]\text{)} \end{aligned}$$

If $v > \overline{v}$, then $Q_a^* \in [z/b, \min(v, \overline{m})/(2b))$ and (72) implies

$$u_s^* = u_s(Q_s^*) \geq u_s(Q_a^*) > u_a(Q_a^*) = u_a^*. \tag{74}$$

Case 2: $\overline{m} = \infty$. We will establish that (30) holds with strict inequality when v is sufficiently large. To do so, we will establish that

$$u_s(Q_a^*) > u_a(Q_a^*) \text{ for } v > \max(4z + c/\overline{F}(2z), \mu - c, \tilde{v}) \tag{75}$$

for some finite $\tilde{v}$. Suppose that $v > \max(4z + c/\overline{F}(2z), \mu - c)$. Because Q_a^* satisfies (25),

$$c - \int_{bQ_a^*}^{\overline{m}} mdF(m) = (v - 4bQ_a^*)\overline{F}(2bQ_a^*) - \int_{bQ_a^*}^{2bQ_a^*} mdF(m). \tag{76}$$

Because $Q_a^* \in [z/b, v/(2b))$, (73) holds when $Q = Q_a^*$. Using this and (76) yields

$$\begin{aligned} u_s(Q_a^*) - u_a(Q_a^*) &= \frac{1}{b}\left[\int_0^{bQ_a^*} mdF(m)\left(v/2 - 2bQ_a^* - \left[(v - 4bQ_a^*)\overline{F}(2bQ_a^*) - \int_{bQ_a^*}^{2bQ_a^*} mdF(m)\right]\Big/\overline{F}(bQ_a^*)\right)\right. \\ &\quad \left. +(v/2 - bQ_a^*)\int_{bQ_a^*}^{2bQ_a^*}(2bQ_a^* - m)dF(m)\right]. \end{aligned}$$

Because Q_a^* satisfies (25), $\lim_{v\to\infty} Q_a^* = v/(4b)$. Therefore,

$$\begin{aligned} \lim_{v\to\infty}\,[u_s(Q_a^*) - u_a(Q_a^*)] &= \lim_{v\to\infty}\frac{1}{b}\left[\int_0^{v/4} mdF(m)\left(\int_{v/4}^{v/2} mdF(m)\Big/\overline{F}(v/4)\right)\right. \\ &\quad \left. +(v/4)\int_{v/4}^{v/2}(v/2 - m)dF(m)\right] > 0. \end{aligned}$$

Because Q_a^* is continuous in v, and $u_s(Q)$ and $u_a(Q)$ are continuous in Q, we conclude that there exists $\tilde{v} < \infty$ such that (75) holds. Inequality (75) implies that for $v > \max(4z + c/\overline{F}(2z), \mu - c, \tilde{v})$, (74) holds.

(b) First, we establish the results for $v \leq \underline{v}$. Let

$$\begin{aligned}\Gamma(Q) &= \int_0^{2bQ-z} m^2/(2b)dF(m) + \int_{2bQ-z}^{z} (3m - 2bQ + z)(m + 2bQ - z)/(8b)dF(m) \\ &\quad + \int_z^{\overline{m}} (m - bQ/2)QdF(m) + e\left[E\min(M, 2bQ - z) + \mu - c\right]/(2b) - cQ.\end{aligned}$$

Suppose $c < \mu$. Then because z is the unique solution to $\int_z^{\overline{m}}(m-z)dF(m) - c = 0$, $\int_x^{\overline{m}}(m-x)dF(m)$ is decreasing in x, and $\int_{\mu-c}^{\overline{m}}(m-\mu+c)dF(m) - c = \int_0^{\mu-c}(\mu - c - m)dF(m) > 0$, $\mu - c < z$. If $v \leq z$, then $a^* = 0$ (as established in Part (a)), which implies

$$w_a^* = \Gamma(\underline{Q}); \tag{77}$$

if $v \leq z$, then $c \leq \int_v^{\overline{m}}(m - v)dF(m) < \min(\mu - (v - z)\overline{F}(z)/2, \mu + v)$, so $Q_s^* < z/b$ (by Proposition 6), which implies

$$w_s^* = \Gamma(Q_s^*). \tag{78}$$

Note that

$$\begin{aligned}(\partial/\partial Q)\Gamma(\underline{Q}) &= \int_0^z (m/2)dF(m) + \int_z^{\overline{m}} (m - z/2)dF(m) + e - c \\ &= \int_0^z (m/2)dF(m) + (z/2)\overline{F}(z) + e > 0,\end{aligned}$$

where the second equality follows because $c = \int_z^{\overline{m}}(m - z)dF(m)$. Because $\Gamma(Q)$ is continuous in Q and because Q_s^* is is continuous and strictly increasing in v on $v \in (\mu - c, z]$ (by Proposition 6), there exists $\underline{v} \in (\mu - c, z]$ such that if $v \in (\mu - c, \underline{v}]$,

$$\Gamma(Q_s^*) > \Gamma(\underline{Q}).$$

This, along with (77) and (78), implies that for $v \in (\mu - c, \underline{v}]$, (31) holds with strict inequality. If $v \leq \mu - c$, then $a^* = s^* = 0$, and (31) holds with equality. Suppose instead that $c \geq \mu$. Then $z = 0$. If $v = 0$, then by Propositions 5 and 6, $a^* = s^* = 0$, and (31) holds with equality.

Second, we establish the results for $v > \overline{v}$. Because Q_a^* satisfies (25) when $v > c - \mu$, Q_s^* satisfies (28) when $v \geq 2(\mu - c)/\overline{F}(z) + z$, and $\overline{m} < \infty$, $\lim_{v\to\infty} Q_a^* = \overline{m}/(2b)$ and $\lim_{v\to\infty} Q_s^* = \overline{m}/b$. Therefore,

$$\lim_{v\to\infty}[w_s^* - w_a^*] = (E[M^2]/4 + e\mu - c\overline{m})/(2b) > 0,$$

where the inequality follows because $e > (c\overline{m} - E[M^2]/4)/\mu$. Because Q_a^* and Q_s^* are continuous in v, w_a^* is continuous in Q_a^*, and w_s^* is continuous in Q_s^*, we conclude that there exists $\overline{v} < \infty$ such that (31) holds with strict inequality if $v > \overline{v}$.■